CFT: QFTs invariant under conformal symmetry.

\( \phi(x) \) scalar field, \( x \in \mathbb{R}^{d-1} \)

Scale transformation \( x' = \lambda x \)

\( \phi'(\lambda x) = \lambda^{-d} \phi(x) \)

\( \Delta = \) scaling dimension of \( \phi \)

\( \Delta = \frac{1}{2} (d-1) \)

\( S[\phi] = \int d^dx \, \partial^\mu \phi \partial_\mu \phi \) is scale invariant

\( S[\phi] = \int d^dx \, (\partial^\mu \phi \partial_\mu \phi - m^2 \phi^2) \) has correct mass dimension but is not scale invariant.

CFTs are scale invariant \( \rightarrow \) no length scale, no mass scale

Classical scale invariance is easily broken at the quantum level.

\( d=4 \) massive \( \phi^4 \)

\( S[\phi] = \int d^dx \left( \frac{1}{4!} \partial^\mu \phi \partial_\mu \phi - \frac{\Delta}{4!} \phi^4 \right) \)

is classically scale invariant \( (\Delta \) dimensionless coupling \)

But a mass term is acquired during renormalization breaking scale invariance.

2D conformal field theory plays a central role in worldsheet dynamics of string theory.

\( \Sigma \) \( \rightarrow \) \( X^\mu(\sigma, \tau) \in \mathbb{R}^{1,0-1} \)

Each \( X^\mu(\sigma, \tau) \) is a \( \phi^\mu \) scalar field in a CFT on \( \Sigma \).
Condensed Matter

Stat Mech: CFTs describe behavior near 2\textsuperscript{nd} order
phase transition

Two natural scales
- size of lattice spacing $a$
- size of object $L$

Corresponding invariant holds true in intermediate scale
$a \ll \Lambda \ll L$

Renormalization group flow to IR (long distance)

CFTs and renormalization group fixed points

Analogies:

CFT in dim $1+(d-1) \leftrightarrow$ Stat. system in dim $d$

$Z = \sum e^{i\mathbf{q} \cdot \mathbf{r}} / k \leftrightarrow Z = \sum e^{-\beta H}$

$1/k \leftrightarrow \beta = 1/\kappa$

- classical limit $\rightarrow$ zero temperature
- quantum fluctuations $\leftrightarrow$ thermal fluctuations
- massive modes $\leftrightarrow$ finite correlation lengths
- massless modes $\leftrightarrow$ infinite correlation length

Math course this
\[ S[\phi] = \frac{1}{2} \int d^d x \left[ \phi \partial_x \phi - m^2 \phi^2 \right] \]

Propagator:
\[ K(x, y) = \langle \phi(x) \phi(y) \rangle = \frac{1}{2} \int d^d \omega \, e^{i \omega y} e^{-i \omega x} \]
\[ = \delta(x-y) \]

To make sense of either
1. Introduce convergence factor \( e^{-\frac{1}{2} \epsilon \phi^2} \)

This leads to Feynman propagator
\[ K(x, y) = \Delta_F(x-y) = \int \frac{d^d p}{(2\pi)^d} \, \frac{e^{-i p \cdot (x-y)}}{p^2 - m^2 + i\epsilon} \]

Particles propagate forward in time
And particles propagate backwards in time

2. Wick rule:
\[ t = x^0 = -i \chi_0 \]
\[ S_E[\phi] = \frac{1}{2} \int d^d x \left( \sum_k (\partial_k \phi_k)^2 + m^2 \phi^2 \right) \]
\[ Z_E = \int \mathcal{D}\phi \, e^{-S_E[\phi]} \]

The Euclidean formulation is more relevant for condensed matter analogies and applications.

In \( d=2 \)
\[ K(x, y) = -\frac{i}{2\pi} \ln |x-y| \quad m=0 \]
\[ K(x, y) \sim e^{-m \ln |x-y|} \quad m > 0 \]

\[ = \int_0^\infty dt \, e^{-m \sin^2 (x-y) / \lambda} \]
Correlation function decay exponentially

\[ \text{correlation length } \sim 1/n \]

\[ S = \int d^d x \mathcal{L}(\Phi, \partial \mu \Phi) \]

\[ \text{EOM } \frac{\delta S}{\delta \Phi} = \partial_\mu \left( \frac{\delta}{\delta \partial_\mu \Phi} \right) \]

(Canonical) stress-energy-momentum tensor

\[ \Theta^{\mu}_{\nu} = \frac{\delta}{\delta \partial_\mu \Phi} \partial_\nu \Phi - \delta^{\mu}_{\nu} \mathcal{L} - T^{\mu}_{\nu} \frac{\delta}{\delta (\partial_\lambda \Phi)} \eta^{\lambda \nu} \]

(4) Canonical change of variables (spacetime and field)

\[ X^\mu \rightarrow X^\mu + \chi^\mu \]

\[ \Phi(x) \rightarrow \hat{\Phi}(x) \]

\[ S \rightarrow \hat{S} = \int d^d x \mathcal{L}(\hat{\Phi}, \partial_\mu \hat{\Phi}, \Phi) \]

Suppose we have a 1-parameter family of inf. symmetry

\[ \hat{X}^\mu = X^\mu + X^\mu \delta \omega \]

\[ \hat{\Phi}(x) = \Phi(x) + \hat{\Phi}(x) \delta \omega \]

\[ \omega \text{ inf. param. } \hat{S}_\omega = S \]

Then

\[ J^\mu_{\Phi} = + \frac{\delta}{\delta (\partial_\mu \Phi)} \hat{\Phi} = \Theta^{\mu}_{\delta} \delta \hat{X}^\delta \]

is conserved:

\[ \partial_\mu J^\mu_{\Phi} = 0 \]

Conserved charge

\[ Q^\mu = \int_{\partial D} J^\mu \]

Exercise: Find conserved current for scaling symmetry

Ex: \[ \Delta x = \epsilon x \]

\[ \Delta \Phi = 0 \]

\[ Q^\mu = \epsilon \Phi^\mu \]

Rotation invariance \rightarrow angular momentum
The transformation on fields is simply generated by

\[ i C_{\mu} \Phi = X^\mu \partial_\mu \Phi - \Phi. \]

So for transformation, \( \Phi \rightarrow \Phi' \), \( X^\mu \rightarrow \delta^\mu_\nu \)

\[ i \delta^{\nu}_{\mu} \Phi = \partial_\mu \Phi. \]

\[ L^\rho_\nu = c(x^\rho \partial_\nu - x^\nu \partial_\rho) + \Phi^\rho_\nu. \]

Scale: \( X^\mu = x^\mu \) & \( \delta^\mu_\nu = \delta_\nu^\mu \)

\[ \partial_\mu \Phi = x^\mu \partial_\mu \Phi. \]

In general, \( \tau \rightarrow S = \int d^4x \sqrt{g} L(g, \Phi, \partial_\nu \Phi) \)

\[ T^\mu_\nu \sim \frac{1}{\sqrt{g}} \frac{\delta S}{\delta g_{\mu\nu}} \] is a stress tensor:

\[ \delta S \sim -\frac{1}{2} \int d^4x T^\mu_\nu \left( \partial_\nu \Phi \right) \Phi \] and \( \Phi^\mu = x^\mu, \Phi^1 = \Phi \nabla \Phi \)

Exercise: check \( T^\mu_\nu \) is a stress tensor.