Recall for a free boson we have the expression

\[ T(z) T(w) \sim \frac{1}{(z-w)^2} + \frac{2T(w)}{(z-w)^2} + \frac{2T(w)}{z-w} \]

More generally, we have in \( \mathbb{C} \) the old

\[ T(z) T(w) \sim \frac{\zeta(2)}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{2T(w)}{z-w} \quad ; \quad \frac{8 \pi \eta^{-1}}{\zeta(1/2)} = \frac{\zeta(2)}{(z-w)^4} \]

\( c \) is called the central charge. By contour integration

\[ \oint C_{\gamma} T(z) = -\frac{c}{12} E''(i) \]

\[ \int_{\gamma} T(z) = \frac{c}{12} \left\{ \text{other terms} \right\} \]

which integrates to

\[ \left( \frac{\partial}{\partial z} \right)^2 T(w) = T(z) - \frac{c}{12} \left\{ \text{other terms} \right\} \]

\[ \left( \frac{\partial}{\partial z} \right)^2 \frac{1}{(z-w)^2} = \frac{c}{12} \left\{ \text{other terms} \right\} \]

\( \text{Hw: Check that this is the correct identity from by working \( y \rightarrow y+y \).} \)

\( \text{Verifying that it gives the right infrared form} \)

\( \text{when} \: w = z + E(z), \: \text{and that the three transformations} \)

\( \text{law of} \: T \: \text{is compatible with successful coordinate transforms} \)

\( \text{in this case.} \)

Note: a general \( SL(2, \mathbb{C}) \) matrix

\[ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \quad ; \quad a \bar{d} - b \bar{c} = 1 \] is in \( SL(2, \mathbb{C}) \) if \( ad - bc = 1 \) and not primitive.

Doing a mode expansion

\[ T(z) = \sum \frac{L_n}{2 \pi i^2} \quad (L_n \text{has} \: \text{and} \: \text{modulus} \: \text{of} \: n) \]

we get by contour arguments

\[ [L_n, L_m] = (n-m) L_{n+m} + \frac{c(n^2 - n)}{12} \delta_{n,-m}. \]
\[ L_n = \frac{1}{2\pi \alpha} \oint_{C_3} dz z^{n-1} T(z) \]

\[ L_m = \frac{1}{2\pi \alpha} \oint_{C_3} dw w^{m-1} T(w) \]

What is the consequence of setting

\[ C_1 = C_3 = \gamma \]

Then conserved charges (on the middle circle):

\[ [L_n, L_m] = \left( \frac{1}{2\pi \alpha} \right)^2 \oint_{C_2} dz \oint_{C_2} dw \ z^{n-1} w^{m-1} T(z) T(w) \]

\[ = \left( \frac{1}{2\pi \alpha} \right)^2 \oint_{C_2} dz \oint_{C_2} dw \ z^{n-1} w^{m-1} T(z) T(w) \]

This is a general way to compute the conserved charges from \( OPE \). We have

\[ z^{n-1} w^{m-1} T(z) T(w) \sim \frac{c/2 \ z^{n-1} w^{m-1}}{(z-w)^4} + \frac{\gamma^{n-1} w^{m-1}}{(z-w)^2} + \frac{\gamma^{n-1} w^{m-1}}{z-w} \]

\[ \text{Res} \sim \frac{c}{2} \cdot \frac{1}{6} \ (n-1) \gamma^{n-1} w^{m-1} + 2 \ (n+1) \gamma^{n-1} w^{m-1} T(w) + w^{m-1} \gamma^{n-1} T(w) \]

Expanding \( T(w) \) +

Doing one more contour integration, we get

\[ [L_n, L_m] = \frac{c}{12} (n^2-n) \delta_{nm} \gamma + 2 (n+1) \gamma^{n-1} L_{nm} - (n+1) \gamma^{n-1} L_{nm} \]

\[ = (n-2) \gamma^{n-1} L_{nm} + \frac{c}{12} (n^2-n) \delta_{nm} \]
Similarly, have modes $\hat{\mathcal{L}}(\hat{\mathcal{L}}) = \sum \frac{\hat{L}_n}{\hat{\mathcal{L}}^{n+1}}$ and

$$\left[ \hat{L}_n, \hat{L}_m \right] = \frac{c}{12} (\hat{\mathcal{L}}^{n+m}) \delta_{n+m,0} + (\hat{\mathcal{L}}^n \hat{\mathcal{L}}^m \right) \hat{L}_{n+m}$$

For this algebra in $\mathcal{L}_n, \mathcal{L}_m$ is the Virasoro algebra.

It operates on the fields in a CFT.

Recall: Really Virasoro algebra exists independent of $c, \hat{c}$. They are just eigenvalues of a central elt.

Suppose $\psi$ and $\phi$ are fields of weight $h$. Here

$$\psi = \sum \frac{\phi}{\hat{\mathcal{L}}^{n+1}}$$

How show that if $\mathcal{L}(\mathcal{L})$ is primary we have

$$\left[ L_n, \phi \right] = \left[ (h - \frac{c}{24}) n - \frac{c}{24} \right] \phi_{n+1}.$$

Insert discussion of $c$ here.

Let's analyze the free boson in modes. While it can be done directly in holomorphic cases, let's instead derive it from a cylinder of current $L$.

$\phi(x,t) \rightarrow x \rightarrow x + L$ and then put $z = \exp(2\pi (z - i\lambda) / L)$

and end to get back to the complex plane.

$$\phi(x,t) = \sum e^{2\pi i n x / L} \phi_n(t)$$

Fourier expand

$$\frac{g}{2} \int \frac{d^2}{2^2} e^{2\pi i x \omega} \frac{d^2}{2^2} \phi = \frac{g L}{2} \sum \left[ \phi_n \phi_{-n} - \left( \frac{2\pi i n}{L} \right)^2 \phi_n \phi_{-n} \right]$$

Conj moments to $\phi_n: \Pi_n = g L \phi_{-n} \left[ \phi_n, \Pi_m \right] = i \delta_{nm}$
What is the physical meaning of $C^2$? Measurements response of the system to length scales (e.g., boundary conditions, metrics).

Ex: Cylinder of radius $R$. $w = R \ln z$ $w = \Sigma B_z$ $v = B_z$ \text{ cylind.}

Since $T_{\mu\nu}$ is normal ordered, $\langle T_{\mu\nu} \rangle = 0$

But $T_{\mu\nu} = \left( \frac{\partial w}{\partial \xi^\lambda} \right)^2 T(w) + \frac{C}{12} \left[ \left( \frac{\partial w}{\partial \xi^\lambda} \right)^2 \right]^{\frac{1}{2}}$

$T(w) = \frac{1}{R^2} \left[ 2 \frac{\partial T(z)}{\partial z} - \frac{C}{2} \right]$

\[ \left\langle T(w) \right\rangle = -\frac{C}{24 R^2} \text{ C.o. mass energy density} \]

(Actually $T_{\mu\nu} \sim T(w)$ so $\left\langle T_{\mu\nu} \right\rangle \Rightarrow \left\langle T_{\mu\nu} \right\rangle = -\frac{C}{24 R^2}$)

Next: genus metre

$S = \int d^2 x \sqrt{g} \ e^\lambda \ e^\lambda d^2 \lambda$

$T_{\mu\nu} = -\frac{\partial}{\partial \xi^\lambda} \frac{\partial S}{\partial g_{\mu\nu}}$ Note: In string theory, the metric $g_{\mu\nu}$ is a dynamical field, so $T^\mu_{\mu} = 0$ is on FSM.

$\left\langle T_{\mu}^\mu \right\rangle$ is a scalar (no on $\Sigma$, a reparametrization

int of $\mu$ of the metric)

Math fact: only int for one expensible in terms of scalar curvatures
Rabled Riemann tensor

\[ T^\mu_{\nu\rho\sigma} = \frac{1}{2} (g^\mu_{\nu\rho}g^\sigma - g^\mu_{\nu\rho}g^\sigma_{\nu\rho}) \]

and \( R \) have grading dimension 2, so

\[ T^\mu_{\nu\rho\sigma} = aR \] for some scalar \( a \).

\[ a = \frac{c}{24\pi} \]

Can also calculate the path integral on \( \Sigma \) using heat kernel method.

Some points:

\[ S = \int d^4x \sqrt{g} g^{\mu\nu} \phi_0 \phi \]

\[ = -\int d^4x \sqrt{g} \phi \left( \frac{1}{2} \phi \left( g^{\mu\nu} \phi \right) \right) \]

This is a Gaussian. \( \Delta \phi \) vs. \( \phi \) structure:

\[ \Delta \phi = \lambda \phi \Rightarrow \int d^4x \sqrt{g} \Delta \phi = \frac{\lambda}{2} \int d^4x \sqrt{g} \phi^2 \]

\[ \Rightarrow S \]

\[ S \geq 0 \quad \{ = 0 \text{ for } \phi \text{ const.} \} = 0 \leq S \]

\[ \lambda > 0 \quad \text{so } \Delta \phi = 0 \text{ const.} \]

\( \Delta \phi = \lambda \phi \)

Mean in field space: \((\phi_0, \phi_2) \)

Expand in modes:

\[ \phi = \sum \phi_n \quad \phi_{(0)} = \phi_0 \quad \lambda \phi \]

\[ \Delta \phi = \pi \phi_n \]

\[ \Rightarrow S = \sum \phi_n \phi_n \]

\[ \Rightarrow Z = \prod \sqrt{\frac{2\pi}{\lambda}} \phi_n \exp(-\Delta) \]

Problem: \( \lambda \) may be non-unique. Renormalize, we have how:

\[ G(x, y) \]

\[ \begin{cases} e^{-y} & \text{if } y > 0 \\ 0 & \text{if } y < 0 \end{cases} \]

\[ G(x, y) = \frac{1}{\sqrt{y_0^2 + \frac{1}{2y_0^4}}} \]
\[ H = \frac{1}{2\lambda} \sum (\bar{\pi}_n \bar{\pi}_m + (2\pi i)^2 \phi_n \phi_m) \quad \bar{\phi}_n = \phi_{-n} \quad \pi_n = \pi_{-n} \]

Decomposed harmonic osc., \( \omega_n = \frac{\omega_0}{\sqrt{n+1}} \).

\[ a_n = -\frac{i}{\sqrt{n+1}} (2\pi \gamma_{n+1} \phi_n + i \pi_n) \quad \bar{a}_n = -\frac{i}{\sqrt{n+1}} (2\pi \gamma_{n+1} \phi_n + i \pi_n) \]
\[ a_{-n} = \frac{i}{\sqrt{n+1}} (2\pi \gamma_{n+1} \phi_{-n} - i \pi_{-n}) \quad \bar{a}_{-n} = \frac{i}{\sqrt{n+1}} (2\pi \gamma_{n+1} \phi_{-n} - i \pi_{-n}) \]

\([a_n, a_m] = \sqrt{n+1} \delta_{nm} \quad [a_n, \bar{a}_n] = 0 \quad [a_n, \bar{a}_{-n}] = \sqrt{n+1} \delta_{nm} \]
\[ a_n^+ = \bar{a}_{-n} \quad \text{Bosonic osc. algebra} \quad [a_n, \pi_n] = \frac{\pi_n}{\sqrt{n+1}} \]

\[ H = \frac{\pi_0^2}{2\lambda} + \frac{i}{\sqrt{n+1}} \sum a_{n} \pi_{n} + \bar{a}_{-n} \bar{\pi}_{n} \]

\[ [H, a_n] = -\frac{2\pi \gamma_n}{\sqrt{n+1}} \quad a_n(t) = a_n(0) e^{-\frac{2\pi \gamma_n t}{\sqrt{n+1}}} \]
\[ \bar{a}_n(t) = \bar{a}_n(0) e^{-\frac{2\pi \gamma_n t}{\sqrt{n+1}}} \]

\[ [H, \pi_0] = 0 \quad \pi_0 \text{ constant} \]

\[ [H, \phi_0] = \frac{\phi_0}{\sqrt{\lambda}} \]

\[ \phi_0(x, t) = \phi_0(t) e^{\frac{2\pi i \chi x}{\sqrt{\lambda}} + \frac{2\pi i \gamma_0 t}{\sqrt{\lambda}}} \]

\[ \gamma = \frac{\lambda}{\sqrt{\lambda}} \sum_{n \neq 0} \frac{1}{\sqrt{n+1}} e^{-\frac{2\pi \gamma_n t}{\sqrt{n+1}}} \]

\[ t = -\infty \quad \text{End. osc.} \]

\[ \text{End. osc.} \quad \sum_{n \neq 0} \frac{1}{\sqrt{n+1}} \quad \exists \text{osc.} \]

\[ \sum_{n \neq 0} \frac{1}{\sqrt{n+1}} \]

\[ \sum_{n \neq 0} \frac{1}{\sqrt{n+1}} \]