The Mathematics of Superconductors

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We face a big challenge in mathematical physics, to explain physical phenomena rigorously from first principles. The cool phenomenon of superconductivity, for instance, poses the question: Can we start from microscopic models and derive rigorously the phenomenological theories of superconductivity? We understand only part of the connection between the microscopic models and the macroscopic equations thought to summarize them.

Superconductivity is a macroscopic phenomenon where certain materials cooled off below a critical temperature (around 5 degrees Kelvin, or -268 degrees Celsius) have essentially zero resistivity, and current can flow freely with no loss. But there is more to superconductivity than just perfect conductivity, namely the Meissner effect, where the magnetic fields passing through the superconductor are expelled and bend around it. This allows levitation and has led to important applications such as the superconducting magnets used in particle accelerators and MRI machines.

Figure 1: Superconductors levitate because magnetic fields bend around them (courtesy of Argonne)

Phenomenological theories of superconductivity include Bardeen, Cooper, and Schrieffer's approximating theory proposed in 1957 at the University of Illinois at Urbana-Champaign; Ginzburg and Landau's theory in 1950 as a macroscopic description for the critical temperature; and Bose-Einstein condensation, in the 1960s, for zero temperature. Physicists are trying to build and understand new superconducting materials that could lead to innovative technologies, including physicist Nadya Mason at Illinois who has discovered surprising zero-temperature metallic states and other cool features in superconductor arrays.

The goal of this project is to study the XY model, a widely used microscopic model of superconductors. We will study this microscopic model and its macroscopic behavior in order to develop mathematical tools that should lead to rigorous connections between microscopic and macroscopic descriptions. In particular, we will discover the critical behavior of the XY model and explore the notion of criticality in the macroscopic equations. This will combine several areas of mathematics (probability, differential equations, and dynamical systems), laying the foundation for a more complete understanding of superconductors from the microscopic first principles, and possibly leading to predictions for new experiments and new materials.

Scaling limits are fundamental

To understand such phenomena, a powerful approach is to construct scaling limits, which can be visualized by considering a drop of red dye in water. If we could zoom in to watch the dye on the molecular scale, the particles would look like billiards hitting each other. Each red dye particle goes in a straight line for a short time, then hits another particle and goes in a new direction until the next collision. The particles' microscopic motion is Newtonian and sensitive to their initial position and velocity. Zooming out to the large scale, we see the dye diffusing throughout the water, spreading out uniformly. The macroscopic behavior is probabilistic and given by the diffusion (or heat) equation. Connecting this microscopic motion to the macroscopic behavior is the kind of scaling limit that we aim to construct.
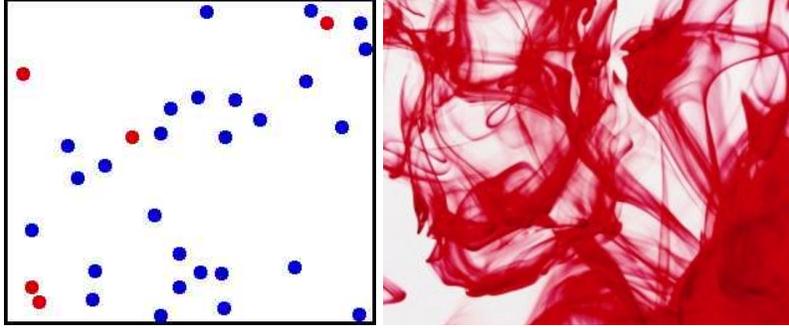
Figure 2: Left: Microscopic dye particles bounce off other particles (courtesy of Greg L, Wikipedia). Right: Red dye diffuses in water on the macroscopic scale (Digital Vision/Getty Images).

Constructing scaling limits requires mathematical tools from probability, functional analysis, and partial differential equations. We develop tools to prove scaling limits from microscopic to macroscopic descriptions and also study the mathematical aspects of both descriptions. One of the most important scaling limits is the mean-field limit, which often gives the first insights into physical phenomena.

There are several important models of superconductors and magnets, called spin models because the particles have magnetic spins restricted to two dimensions (the XY model of superconductors) or three dimensions (the Heisenberg model of magnets). In recent work with E. Meckes, we have taken the mean-field scaling limit approach to the Heisenberg model and discovered interesting non-normal behavior at the critical temperature, the crucial transition between two different kinds of behavior. Below the critical temperature, spins tend to align; above the critical temperature, spins tend to be independent and do not align. Both above and below the critical temperature, the total spin has Gaussian limiting distributions. It is only at the critical transition temperature that the limiting distribution is non-Gaussian.

Macroscopic behavior of the XY model

The first goal of this project is to discover the critical behavior of the XY model, similar to the Heisenberg theorem above, but with a new unknown limiting distribution. This will be aided by simulating the mean-field XY model to help determine the scaling factor and the form of the non-Gaussian limiting distribution.

We have also been studying the phenomenon of metastability in the two-dimensional XY model on a periodic lattice, a phenomenon that is important because the typical superconducting state observed in the laboratory is a metastable state and not the ground state–the state which is the absolute lowest on the energy landscape, which is analogous to the Dead Sea shore as the lowest on the Earth. Metastable states (see the next figure for an example) have the lowest energy only locally, analogous to valleys, even high mountain valleys. With undergraduate student Jack Weinstein, we have identified previously unknown metastable states in the XY model with a magnetic field. With Lee DeVille, we will study the dynamics between metastable states in the XY model as they pass through saddle points, analogous to mountain passes.

Another goal of this project is to discover the feature of the macroscopic equations corresponding to metastability in the XY model, as well as the feature corresponding to the critical behavior of the XY model and its non-Gaussian limiting distribution. This will help lay the foundation for a rigorous scaling limit from the microscopic XY model to the macroscopic equations–and a more complete understanding of superconductors.
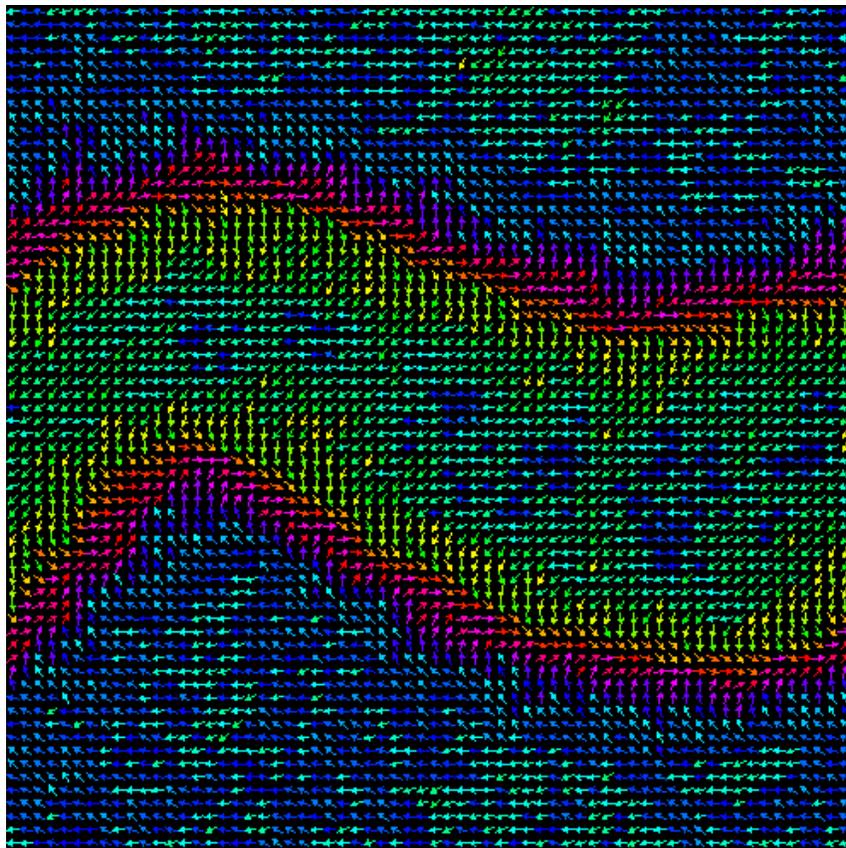
Figure 3: Magnetic spins (arrows color-coded to direction) tend to align locally in the XY model at low temperatures, but there can be macroscopic defects as in this metastable state (Jack Weinstein).

Methods and connections to larger research program

In order to understand the mean-field scaling limit of the XY model, we will have to develop an appropriate way of scaling the total spin, similar to the Heisenberg model. We will have a circular symmetry for the spins instead of a spherical symmetry, which will make the calculations specific to this setting, but the general approach will be through Stein's method, a powerful method in probability for proving limit theorems. Stein's method requires a so-called exchangeable pair that we will construct from the Glauber dynamics, a natural dynamics in which one picks a spin at random and re-samples it according to the local magnetic field of the other spins around it. We will also develop a new abstract Stein's theorem for the approximating distribution, which gives a natural way of identifying the non-Gaussian limiting density. This project has grown out of my research program funded by an NSF CAREER award, confirming the existence of metastability in the XY model and a two-step phase transition to superconductivity in coupled XY models as observed in experiments.

My larger research program is focused on developing tools to prove scaling limits for a variety of physical and biological phenomena, deepening our understanding of nonlinear and nonlocal equations, and broadening applications in engineering and medicine. With G. Ben Arous, B. Schlein, and G. Staffilani, we have been working on other mean-field limits, from microscopic quantum dynamics to the macroscopic cubic nonlinear Schrödinger equation (NLS). Another important scaling limit is the continuum limit, which justifies discrete approximations of continuous phenomena. With E. Lenzmann and G. Staffilani, we have studied the continuum limit of nonlinear lattices with long-range interactions, rigorously justifying fractional-derivative NLS models proposed in the physics literature to understand electron transport in biopolymers like DNA. Aiming to rigorously understanding physical and biological phenomena will help us advance mathematical knowledge; and developing mathematical tools will help us understand these phenomena.