Test 3 Review Problems - Solutions.

1. Let $t=$ time in min., $r(t)=$ radius in in., $V(t)=$ volume in in^3.

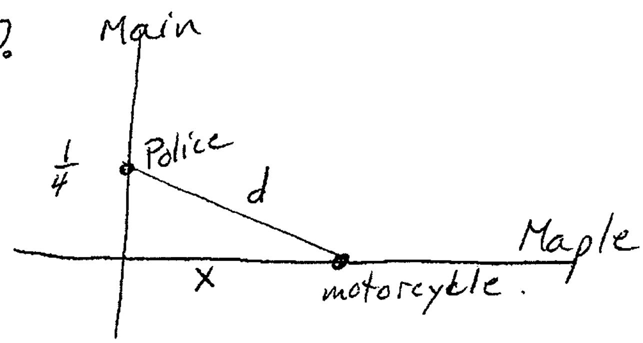
Given: $V'(t)=10$.
Find: $r'(t)$ when $r(t)=5$.

$$V(t)=\frac{4}{3}\pi(r(t))^3.$$

$$V'(t)=4\pi(r(t))^2\,r'(t).$$

$$10=4\pi(5)^2\,r'(t)$$

$$\boxed{r'(t)=}\ \frac{10}{100\pi}=\frac{1}{10\pi}\ \text{in./min.}$$

2. 

Let $x(t)=$ distance of motorcycle from intersection, in miles.

Let $d(t)=$ distance of motorcycle from police car, in miles.

Let $t=$ time in hours.

Given $x'(t)=-40$

want $d'(t)$ when $x(t)=1/8$

when $x(t)=1/8$, $d(t)=\sqrt{\frac{1}{16}+\frac{1}{64}}=\sqrt{\frac{5}{64}}=\frac{\sqrt{5}}{8}$

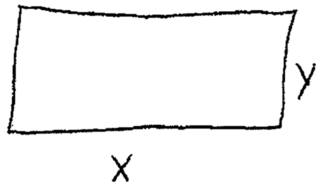
$$\frac{1}{16}+(x(t))^2=(d(t))^2$$

$$2(x(t))\,x'(t)=2\,d(t)\,d'(t)$$

$$2\cdot\frac{1}{8}(-40)=2\cdot\frac{\sqrt{5}}{8}\,d'(t).$$

$$d'(t)=\frac{-40}{\sqrt{5}}\ \text{miles/hr.}$$

3.



Let t = time in sec.

$x(t)$ = base in cm

$y(t)$ = height in cm.

$A(t)$ = area in cm^2

Given

$x'(t) = 4$

$y'(t) = -3$

Want $A'(t)$ when $x = 20$, $y = 12$.

$$A(t) = x(t)\,y(t)$$

$$A'(t) = x'(t)\,y(t) + x(t)\,y'(t)$$

$$A'(t) = 4 \cdot 12 + 20(-3)$$

$$= 48 - 60 = -12$$

The area is <u>decreasing</u> at 12 cm^2/sec.

4\. a. $\int x^{1/2}\,dx = \frac{2}{3}x^{3/2} + C$ ← don't forget $+C$

b. $\int \csc^2\theta\,d\theta = -\cot\theta + C$

c. $\int x^{-1}\,dx = \ln|x| + C$

d. $u = e^{-t}$ $du = -e^{-t}\,dt$ $-2\int du = -2u + C = \boxed{-2e^{-t} + C}$

e. $\boxed{\begin{array}{l} u = 1 - x^2 \\ du = -2x\,dx \end{array}}$ $\int x(1-x^2)^{1/3}\,dx = \frac{-1}{2}\int -2x\,(1-x^2)^{1/3}\,dx$

$$= \frac{-1}{2}\int u^{1/3}\,du = \frac{-1}{2}\cdot\frac{3}{4}u^{4/3} + C$$

$$= \boxed{\frac{-3}{8}(1-x^2)^{4/3} + C}$$

f. $\displaystyle\int \frac{1}{x^2+1}\,dx = \tan^{-1}x + C$

g. $\boxed{\begin{array}{l} u = x^2+1 \\ du = 2x\,dx \end{array}}$ $\displaystyle\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{2x}{x^2+1}\,dx$

$\displaystyle = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C$

$\displaystyle = \frac{1}{2}\ln(x^2+1) + C$

h. This one is a little tricky.

$\boxed{\begin{array}{l} u = x^2 \\ du = 2x\,dx \end{array}}$ $\displaystyle\int \frac{x}{x^4+1}\,dx = \frac{1}{2}\int \frac{2x}{(x^2)^2+1}\,dx$

$\displaystyle = \frac{1}{2}\int \frac{1}{u^2+1}\,du = \frac{1}{2}\tan^{-1}u + C = \frac{1}{2}\tan^{-1}(x^2) + C$

i. $\displaystyle\int \frac{x^4+1}{\sqrt{x}}\,dx = \int \frac{x^4}{\sqrt{x}} + \frac{1}{\sqrt{x}}\,dx = \int x^{7/2} + x^{-1/2}\,dx$

$\displaystyle = \cancel{\frac{2}{9}x^{9/2} + \frac{2}{3}x^{3/2} + C} = \frac{2}{9}x^{9/2} + 2x^{1/2} + C$

j. $\displaystyle\int 3\cos x + 4\sin x\,dx = 3\sin x - 4\cos x + C$

5. $\sum_{i=2}^{4} (2^i + 3i) = (2^2 + 3\cdot 2) + (2^3 + 3\cdot 3) + (2^4 + 3\cdot 4)$

$= 4 + 6 + 8 + 9 + 16 + 12 = 55$

6. $\sum_{i=1}^{100} (4 + 3i - i^2) = \sum_{i=1}^{100} 4 + 3\sum_{i=1}^{100} i - \sum_{i=1}^{100} i^2$

$= 4\cdot 100 + 3\cdot \frac{100(101)}{2} - \frac{100(101)(201)}{6}$

$\cancel{\neq 1357{,}9001}$ $\cancel{= 338{,}1350}$ $-322{,}800$

7. $\sum_{i=1}^{n} \frac{1}{n}\left[\left(\frac{i}{n}\right)^2 - 5\left(\frac{i}{n}\right)\right] = \frac{1}{n}\left[\frac{1}{n^2}\sum_{i=1}^{n} i^2 - \frac{5}{n}\sum_{i=1}^{n} i\right]$

$= \frac{1}{n}\left[\frac{1}{n^2}\frac{n(n+1)(2n+1)}{6} - \frac{5}{n}\cdot\frac{n(n+1)}{2}\right]$

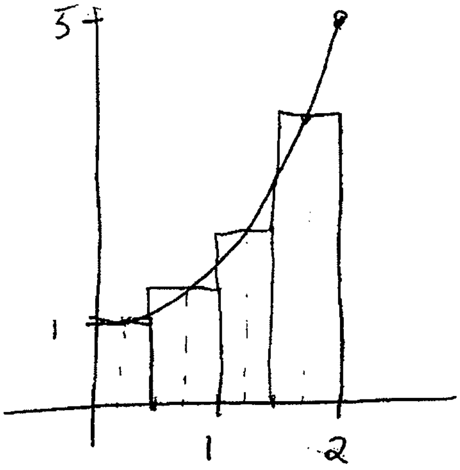
$= \frac{2n^2+3n+1}{6n^2} - \frac{5n+5}{2n}.$

$\lim_{n\to\infty}\left(\frac{2n^2+3n+1}{6n^2} - \frac{5n+5}{2n}\right) = \frac{1}{3} - \frac{5}{2} = \frac{2-15}{6} = \frac{-13}{6}$

<u>Bonus</u> $\int_0^1 x^2 - 5x\,dx = \frac{1}{3}x^3 - \frac{5}{2}x^2\Big|_0^1 = \frac{1}{3} - \frac{5}{2} = \frac{-13}{6}$

I chose $a=0$, $b=1$ so that
$x_i = \frac{i}{n}$, which appears in the sum, and
$\Delta x = \frac{1}{n}.$

8. $a=0,\ b=2,\ n=4$



$$\Delta x = \frac{2-0}{4} = \frac{1}{2}$$

$$x_0=0,\ x_1=\frac{1}{2},\ x_2=1,\ x_3=\frac{3}{2},\ x_4=2$$

$$c_i = \text{midpoint of } [x_{i-1}, x_i]$$
$$= \frac{1}{2}(x_{i-1}+x_i)$$

$$\text{Riemann Sum} = \sum_{i=1}^{4} f(c_i)\Delta x$$

$$= f\left(\tfrac{1}{4}\right)\cdot\tfrac{1}{2} + f\left(\tfrac{3}{4}\right)\cdot\tfrac{1}{2} + f\left(\tfrac{5}{4}\right)\cdot\tfrac{1}{2} + f\left(\tfrac{7}{4}\right)\cdot\tfrac{1}{2}$$

$$= \frac{1}{2}\left[\left(\frac{1}{16}+1\right)+\left(\frac{9}{16}+1\right)+\left(\frac{25}{16}+1\right)+\left(\frac{49}{16}+1\right)\right]$$

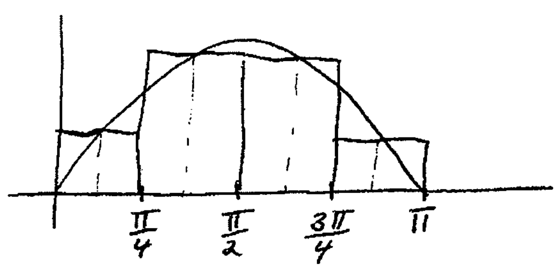
$$= \frac{1}{2}\left[\frac{84}{16}+4\right] = \frac{37}{8}$$

9. $f(x)=\sin x,\ a=0,\ b=\pi,\ n=4$

$$\Delta x = \frac{\pi-0}{4} = \frac{\pi}{4}$$

$x_0=0,\ x_1=\pi/4,\ x_2=\pi/2,$

$x_3=\frac{3\pi}{4},\ x_4=1$



c_i = midpoint.

$$c_1=\pi/8,\ c_2=\frac{3\pi}{8},\ c_3=\frac{5\pi}{8},\ c_4=\frac{7\pi}{8}$$

$$\text{Riemann sum} = \frac{\pi}{4}\left(\sin\left(\frac{\pi}{8}\right)+\sin\left(\frac{3\pi}{8}\right)+\sin\left(\frac{5\pi}{8}\right)+\sin\left(\frac{7\pi}{8}\right)\right)$$

≈ 2.05 (using calculator)

10. $\displaystyle\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x$

where $\Delta x = \dfrac{b-a}{n}$

$\cancel{x_0}$ $x_0, x_1, \ldots, x_n$ is a partition of $[a,b]$ into n equal pieces.

c_i is a point in $[x_{i-1}, x_i]$.

11. $a=1, \quad b=3, \quad f(x)=x^2-x$

$\Delta x = 2/n. \qquad x_i = a + i\,\Delta x = 1 + \dfrac{2i}{n}$

choose $c_i = x_i$

$$\sum_{i=1}^{n} f(c_i)\,\Delta x = \frac{2}{n}\sum_{i=1}^{n} f\left(1+\frac{2i}{n}\right)$$

$$= \frac{2}{n}\sum_{i=1}^{n}\left[\left(1+\frac{2i}{n}\right)^2 - \left(1+\frac{2i}{n}\right)\right]$$

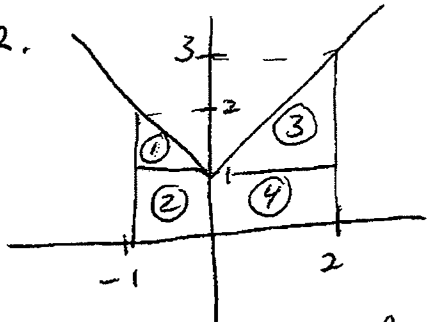
$$= \frac{2}{n}\sum_{i=1}^{n}\left(1+\frac{4i}{n}+\frac{4i^2}{n^2}\right)-\left(1+\frac{2i}{n}\right)$$

$$= \frac{2}{n}\sum_{i=1}^{n}\left[\frac{2i}{n}+\frac{4i^2}{n^2}\right]$$

$$= \frac{2}{n}\left[\frac{2}{n}\sum_{i=1}^{n} i + \frac{4}{n^2}\sum_{i=1}^{n} i^2\right] = \frac{2}{n}\left[\frac{2}{n}\,\frac{n(n+1)}{2} + \frac{4}{n^2}\,\frac{n(n+1)(2n+1)}{6}\right]$$

$$= \frac{2n+1}{n} + \frac{4}{3}(2n^2+3n+1)$$

$$\lim_{n\to\infty}\left[\frac{2n+1}{n}+\frac{2(2n^2+3n+1)}{3}\right] = 2+\frac{8}{3} = \frac{14}{3}$$

12. 

Split the area into rectangles and triangles as shown.

$$\text{Area} = ① + ② + ③ + ④$$
$$= \tfrac{1}{2}\cdot 1\cdot 1 + 1\cdot 1 + \tfrac{1}{2}\cdot 2\cdot 2 + 2\cdot 1$$
$$= \tfrac{1}{2} + 1 + 2 + 2 = 5\tfrac{1}{2}$$

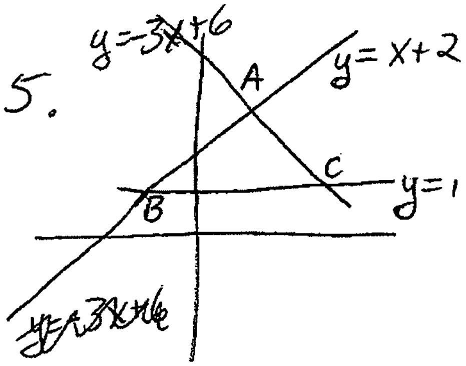
13. ave value $= \dfrac{1}{\pi/2}\displaystyle\int_0^{\pi/2} \cos(2x)\,dx$

$u = 2x \quad du = 2dx \quad$ when $x=0, u=0$, when $x=\pi/2$, $u=\pi$

$$= \frac{2}{\pi}\,\frac{1}{2}\int_0^{\pi/2} \cos(2x)\;2dx = \frac{1}{\pi}\int_0^{\pi} \cos u\, du$$

$$= \frac{1}{\pi}\sin u\Big|_0^{\pi} = \frac{1}{\pi}(\sin\pi - \sin 0) = \frac{1}{\pi}(0-0) = 0.$$

14. $f'(x) = (x^2 3^{(x^2)})\,2x$ so $f'(2) = \cancel{72}\ 4\cdot 3^4\cdot 4 = 1296$

15. 

Find points of intersection

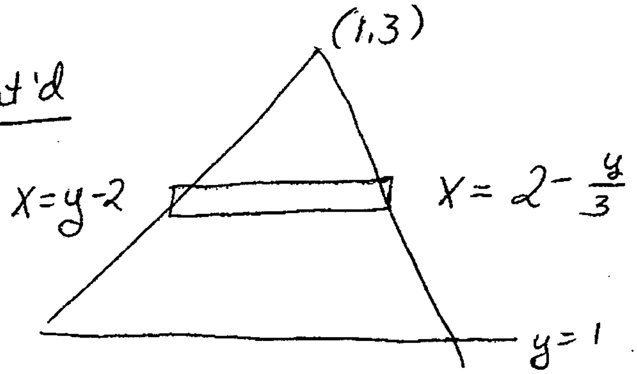
A. $x+2 = -3x+6 \quad 4x=4 \quad x=1$, $y=3$

B. $x+2=1 \quad x=-1 \quad y=1$

C. $-3x+6=1 \quad x=\frac{5}{3} \quad y=1$

$\rightarrow$

15. cont'd

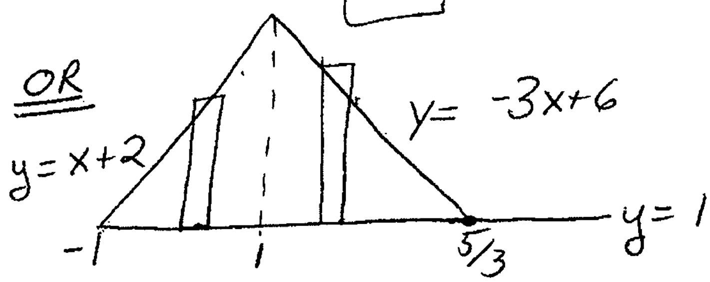


$$\text{area} = \int_1^3 \left(2-\frac{y}{3}\right)-(y-2)\,dy = \int_1^3 4-\frac{4}{3}y\,dy$$

$$= 4y - \frac{2}{3}y^2 \Big|_1^3 = 12-6-\left(4-\frac{2}{3}\right)$$

$$= 2+\frac{2}{3} = \boxed{\frac{8}{3}}$$

OR



$$\text{area} = \int_{-1}^1 (x+2-1)\,dx + \int_1^{5/3} (-3x+6-1)\,dx$$

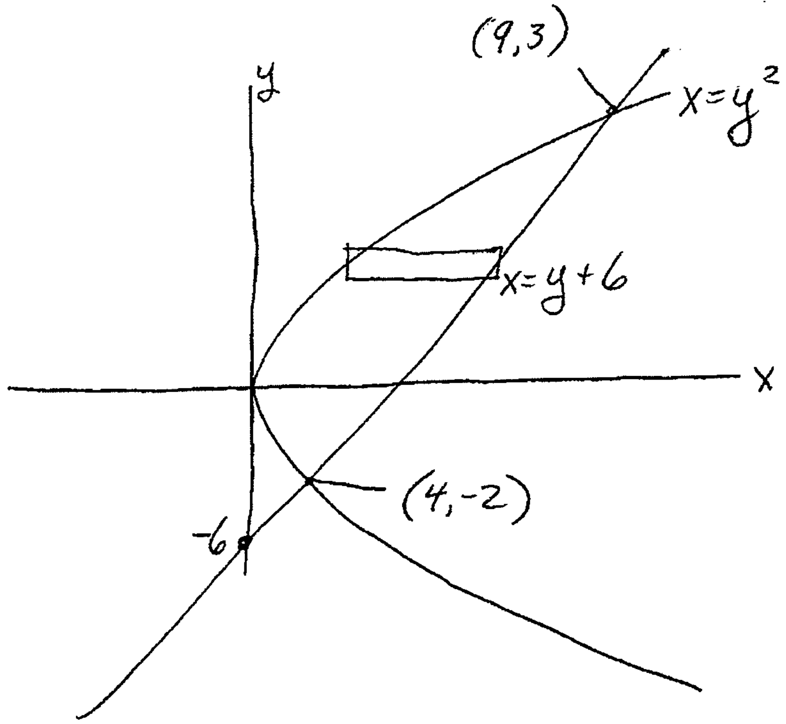
$$= \int_{-1}^1 (x+1)\,dx + \int_1^{5/3} (-3x+5)\,dx$$

$$= \frac{1}{2}x^2+x\Big|_{-1}^1 + \left(-\frac{3}{2}x^2+5x\right)\Big|_1^{5/3}$$

$$= \frac{1}{2}+1-\left(\frac{1}{2}-1\right)+\left(-\frac{3}{2}\left(\frac{5}{3}\right)^2+5\left(\frac{5}{3}\right)\right)-\left(-\frac{3}{2}+5\right)$$

$$= \boxed{\frac{8}{3}}$$

16.



Find points of intersection $y^2=y+6$

$y^2-y-6=0$; $(y-3)(y+2)=0$; $y=3,-2$

when $y=3$, $x=9$, when $y=-2$, $x=4$.

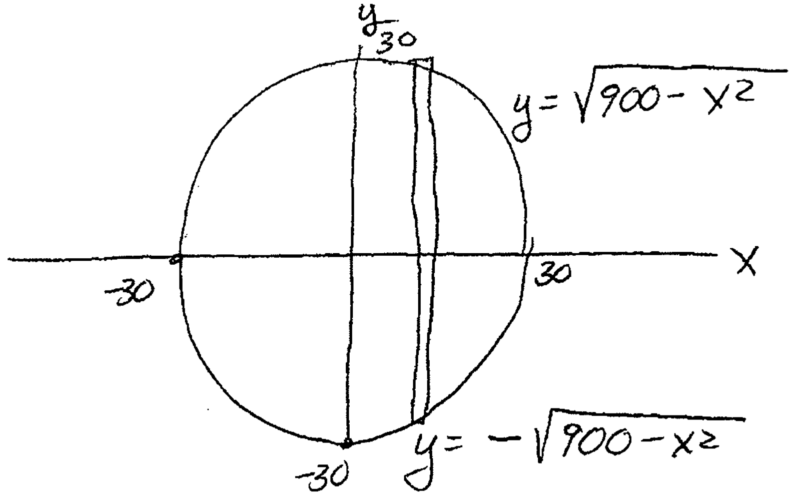
$$\text{area} = \int_{-2}^{3} (y+6)-y^2 \, dy$$

$$= \tfrac{1}{2}y^2+6y-\tfrac{1}{3}y^3 \Big|_{-2}^{3}$$

$$= \tfrac{9}{2}+18-9-\left(2-12+\tfrac{8}{3}\right)$$

$$= \frac{125}{6}$$

17.



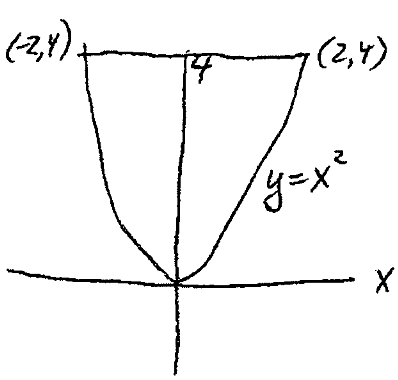
$$A(x) = \left(\sqrt{900-x^2} + \sqrt{900-x^2}\right)^2 = 4(900-x^2)$$

$$\text{Volume} = \int_{-30}^{30} 4(900-x^2)\,dx = 4\left(900x - \frac{x^3}{3}\right)\Big|_{-30}^{30}$$

$$= 4\left(27{,}000 - \frac{27{,}000}{3} - \left(-27{,}000 + \frac{27{,}000}{3}\right)\right)$$

$$= 4(54{,}000 - 18{,}000) = 4 \cdot 36{,}000 = 144{,}000 \text{ ft}^3$$

18. 5.2/26a

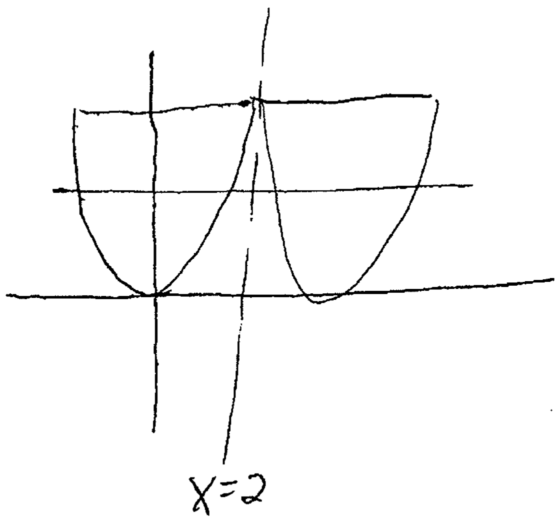


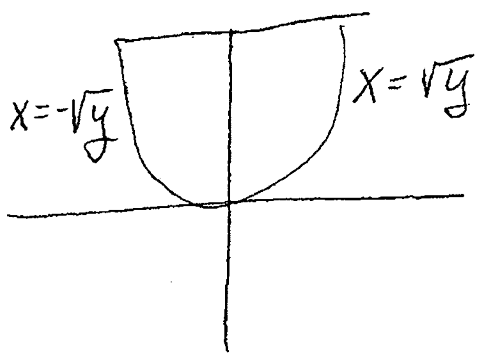
$$\text{Vol} = \int_{-2}^{2} \pi (x^2)^2\,dx$$

$$= \frac{\pi}{5} x^5 \Big|_{-2}^{2} = \frac{\pi}{5}(32 - (-32))$$

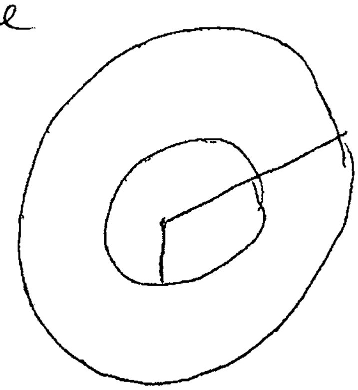
$$= \frac{64\pi}{5}$$

18. cont'd (26e)





Slice



outer radius $= 2+\sqrt{y}$

inner radius $= 2-\sqrt{y}$

$$\text{Vol} = \int_0^4 \pi(2+\sqrt{y})^2 - \pi(2-\sqrt{y})^2 \, dy$$

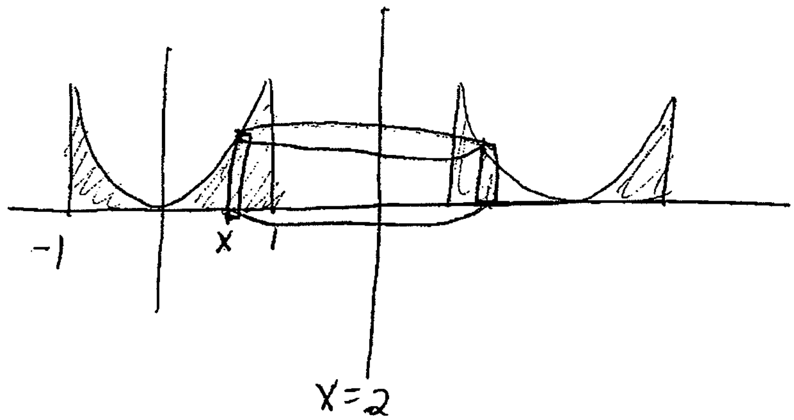
$$= \pi\int_0^4 4+4y^{1/2}+y-(4-4y^{1/2}+y)\,dy$$

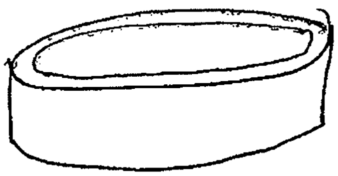
$$= \pi\int_0^4 8y^{1/2}\,dy = 8\pi\,\tfrac{2}{3}\,y^{3/2}\Big|_0^4$$

$$= \frac{16\pi}{3}\left(4^{3/2}-0^{3/2}\right) = \frac{16\pi}{3}\cdot 8 = \frac{128\pi}{3}$$

$^4 16$
$\underline{8}$
128

19.





height $= y = x^2$

radius $= 2-x$

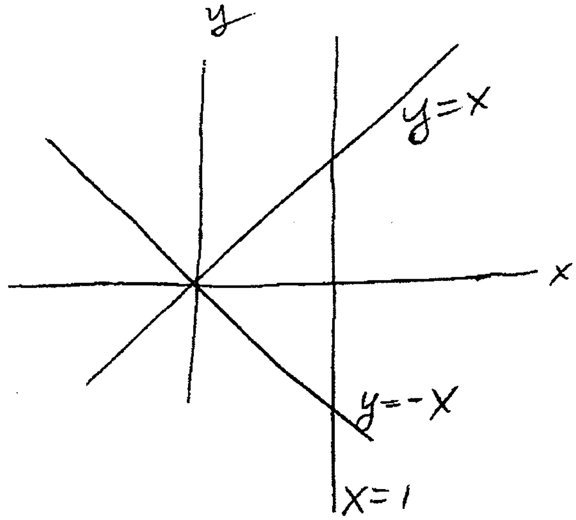
$$\text{Vol.} = \int_{-1}^{1} 2\pi(2-x)x^2\,dx$$

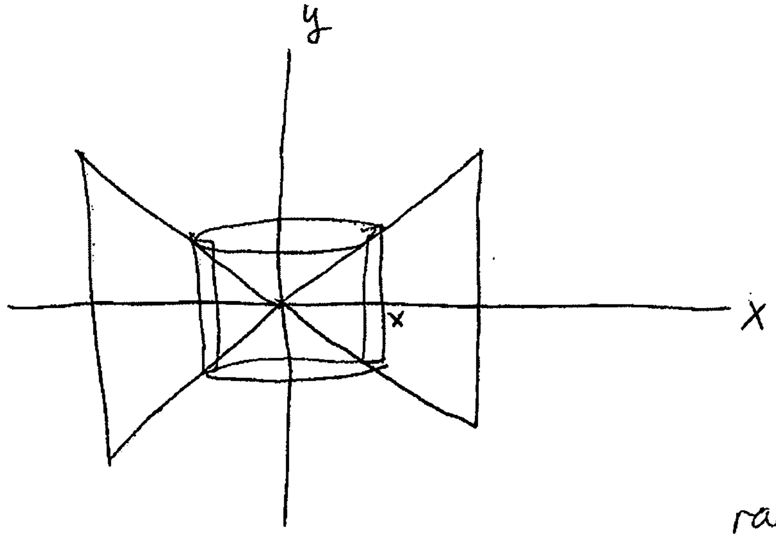
$$= 2\pi\int_{-1}^{1} 2x^2 - x^3\,dx = 2\pi\left(\frac{2}{3}x^3 - \frac{1}{4}x^4\right)\Big|_{-1}^{1}$$

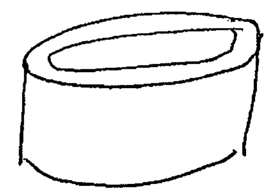
$$= 2\pi\left(\frac{2}{3} - \frac{1}{4} - \left(-\frac{2}{3} - \frac{1}{4}\right)\right)$$

$$= 2\pi\cdot\frac{4}{3} = \frac{8\pi}{3}$$

20.







height $= y_2 - y_1$

$= x-(-x) = 2x$

radius $= x$

$$\text{Vol} = \int_0^1 2\pi x(2x)\,dx$$

$$= \cancel{2}\, 4\pi \int_0^1 x^2\,dx = \frac{4\pi}{3} x^3 \Big|_0^1 = \frac{4\pi}{3}$$

21. a. F

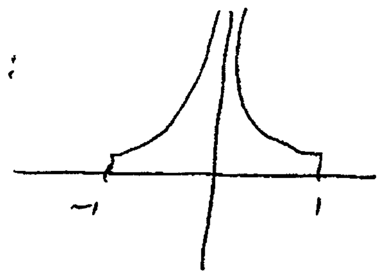
b. T $F(x) = \int_a^x f(t)\,dt$ FTC, Part II

c. T (think of signed area - there is no area below the x-axis)

21 cont'd

d. F. FTC Part I applies only if the function is continuous.

Look at the graph:



If $\int_{-1}^{1} \frac{1}{x^2}\,dx$

is defined at all, it would certainly be positive, not -2. In fact it is $+\infty$, as you will learn in Calc II.

e. T

f. T. See the Integral Mean Value Theorem.

g. F. ~~f(x) is integrab~~ $f(x) = |x|$ is cont., therefore integrable, but it is not diff. at $x=0$.

h. T by integral properties, Section 4.4

i. T See Section 4.4

j. T

k. F - you cannot factor an x past $\int$ sign.

l. F - this just doesn't work. Try any example to see this.

22. See Section 4.5