Please email Dr. Mortensen if you find errors in solutions!

Test 2 Review Solutions

2a. implicit differentiation

$$e^y + xe^y \frac{dy}{dx} - 3\frac{dy}{dx}\sin x - 3y\cos x = 0$$

$$(xe^y - 3\sin x)\frac{dy}{dx} = 3y\cos x - e^y$$

$$\frac{dy}{dx} = \frac{3y\cos x - e^y}{xe^y - 3\sin x}$$

2b. $f'(x) = \dfrac{1}{(x^2-x)^2+1}(2x-1)$

2c. $f'(x) = e^{\sin^{-1}x}\dfrac{1}{\sqrt{1-x^2}}$

5. $f'(x) = \frac{1}{3}(x+1)^{-2/3}$. $\quad f'(0) = \frac{1}{3}$ and $f(0) = 1$.

The linear approx. is $y = 1 + \frac{1}{3}(x-0)$

6. Use $f(x) = \sqrt[3]{x} = x^{1/3}$ and $x_0 = 27$.

Then $f'(x) = \frac{1}{3}x^{-2/3}$ so $f'(x_0) = f'(27) = \frac{1}{27}$; $\quad f(x_0) = 3$

The linear approx. is $y = 3 + \frac{1}{27}(x-27)$.

For the approximation to $\sqrt[3]{28}$, plug in $x = 28$:

$y = 3 + \frac{1}{27}(28-27) = 3 + 1/27$. $\quad \sqrt[3]{28} \approx 3\frac{1}{27}$.

7. Use the function $f(x) = x^3 - 28$.

Then $f'(x) = 3x^2$.

A reasonable starting point is $x_0 = 3$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 3 - \frac{-1}{27} = 3\frac{1}{27}.$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 3\frac{1}{27} - \frac{(3\frac{1}{27})^3 - 28}{3(3\frac{1}{27})^2} \text{ etc.}$$

9a. Type $\frac{\infty}{\infty}$ $\quad \lim_{x\to\infty} \frac{\ln x}{x^{1/2}} = \lim_{x\to\infty} \frac{x^{-1}}{\frac{1}{2}x^{-1/2}}$

$= \lim_{x\to\infty} \frac{2}{x^{1/2}} = 0$

9b. Type $\frac{0}{0}$. $\quad \lim_{x\to 0} \frac{\sin x - x}{x^2} = \lim_{x\to 0} \frac{\cos x - 1}{2x}$

Still type $\frac{0}{0}$. $\quad = \lim_{x\to 0} \frac{-\sin x}{2} = \frac{0}{2} = 0$

9c. $\lim_{x\to\infty} \frac{x^2-1}{x^{-1}-1}$ is <u>not</u> an indeterminate form.

Numerator $\to \infty$, denominator $\to -1$, So limit $= -\infty$.

9d. Indeterminate form type $\infty \cdot 0$. Make it a quotient

$\lim_{x\to\infty} x e^{-x} = \lim_{x\to\infty} \frac{x}{e^x}$ (type $\frac{\infty}{\infty}$) $= \lim_{x\to\infty} \frac{1}{e^x} = 0$.

9e. $\lim_{x\to 3} e^x x^{-1} = \frac{e^3}{3}$ not indeterminate form.

9f. Indeterminate form type $1^{-\infty}$.

$\lim_{x\to 3^+} (x-2)^{\ln(x-3)} = \lim_{x\to 3^+} \exp \ln (x-2)^{\ln(x-3)}$

$= \exp \lim_{x\to 3^+} \ln(x-3) \ln(x-2)$. $\quad$ Indet. form type $(-\infty)\cdot 0$

$= \exp \lim_{x\to 3^+} \frac{\ln(x-3)}{[\ln(x-2)]^{-1}}$ type $\frac{-\infty}{\infty}$

$= \exp \lim_{x\to 3^+} \frac{(x-3)^{-1}}{-[\ln(x-2)]^{-2}(x-2)^{-1}} = \exp \left[\underbrace{\lim_{x\to 3^+} (x-2)}_{=1} \right] \left[\underbrace{\lim_{x\to 3^+} \frac{-[\ln(x-2)]^2}{(x-3)}}_{\text{type } \frac{0}{0}} \right]$

$= \exp \lim_{x\to 3^+} \frac{-2[\ln(x-2)] \frac{1}{x-2}}{1} = \exp 0 = e^0 = 1$

9g. $\lim\limits_{x\to 0} \frac{1}{x^3} - \frac{1}{x^4} = \lim\limits_{x\to 0} \frac{x-1}{x^4}$. Not indeterminate.

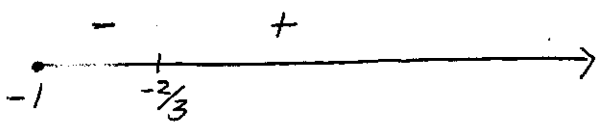
as $x \to 0$, $x-1 \to -1$ and $x^4 \to 0$ and x^4 is positive, so $\lim = -\infty$.

10. $f'(x) = 2(x+1)^{1/2} + 2x \cdot \frac{1}{2}(x+1)^{-1/2} \cdot 1$

$$= 2(x+1)^{1/2} + \frac{x}{(x+1)^{1/2}} = \frac{2(x+1) + x}{(x+1)^{1/2}} = \frac{3x+2}{(x+1)^{1/2}}.$$

Critical points $x = -1$ and $x = -2/3$.

First derivative test



Note: $f(x)$ is undefined for $x < -1$.

Local min occurs at $x = -2/3$. The local min is

$f(-\frac{2}{3}) = 2(-\frac{2}{3})\sqrt{\frac{1}{3}} = \frac{-4}{3\sqrt{3}}$ -3/5 Can also use 2nd deriv. test.

11. $f'(x) = 2\left(x^{2/5} - 3x^{1/5}\right)\left(\frac{2}{5}x^{-3/5} - \frac{3}{5}x^{-4/5}\right)$

$f'(x) = 0$ when $x^{2/5} - 3x^{1/5} = 0$ or $\frac{2}{5}x^{-3/5} - \frac{3}{5}x^{-4/5} = 0$

$x^{2/5} - 3x^{1/5} = 0$

$x^{2/5} = 3x^{1/5}$

$x^2 = 3^5 x$

$x = 3^5 = 243$

$\frac{2}{5}x^{-3/5} - \frac{3}{5}x^{-4/5} = 0$

$\frac{2}{5}x^{-3/5} = \frac{3}{5}x^{-4/5}$ Multiply by $x^{4/5}$

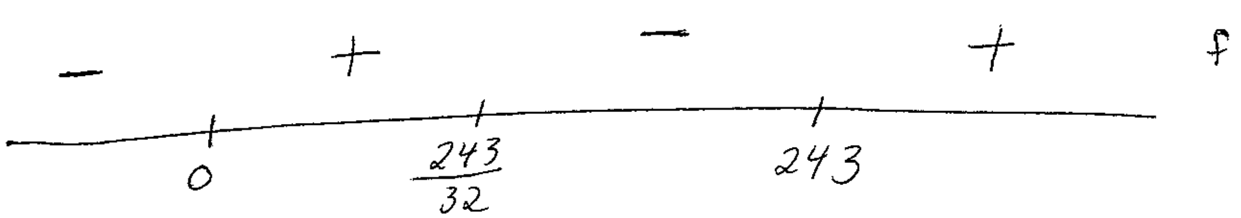
$\frac{2}{5}x^{1/5} = \frac{3}{5}$

$x^{1/5} = \frac{3}{2}$ $x = \left(\frac{3}{2}\right)^5 = \frac{243}{32}$

Also, $f'(x)$ is undefined when $x=0$, because of the negative exponents.

$f'(x)$ will be a little easier to work with if we write it this way:

$f'(x) = \frac{2}{5}\left(x^{\frac{2}{5}} - 3x^{1/5}\right)x^{-4/5}\left(2x^{1/5} - 3\right)$



First derivative test.

Local min at $x=0$ and at $x=243$

Local max at $x = 243/32$.

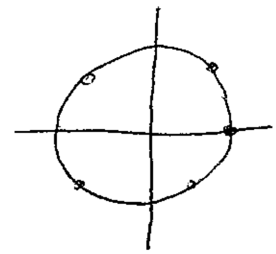
This one is a little hard for an exam! Practice others from Sections 3.4, 3.5, 3.6 also.

12. $f(x) = \sin x \cos x$

$f'(x) = \cos^2 x - \sin^2 x = 0$ for $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

Check critical points and endpoints

x	$\frac{\pi}{4}$	$\frac{3\pi}{4}$	$\frac{5\pi}{4}$	$\frac{7\pi}{4}$	0	2π
$f(x)$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	0	0



The absolute max is $\frac{1}{2}$ and occurs at $x = \frac{\pi}{4}, \frac{5\pi}{4}$

The abs. min. is $-\frac{1}{2}$ and occurs at $x = \frac{3\pi}{4}, \frac{7\pi}{4}$

$+ \quad 0 \quad - \quad 0 \quad + \quad 0 \quad - \quad 0 \quad +$ $\quad f'$

$0 \quad \frac{\pi}{4} \quad \frac{3\pi}{4} \quad \frac{5\pi}{4} \quad \frac{7\pi}{4} \quad 2\pi$

increasing on $(0, \frac{\pi}{4})$ and $(\frac{3\pi}{4}, \frac{5\pi}{4})$ and $(\frac{7\pi}{4}, 2\pi)$

decreasing on $(\frac{\pi}{4}, \frac{3\pi}{4})$ and on $(\frac{5\pi}{4}, \frac{7\pi}{4})$

13. $f(x) = e^{-x} \sin x$.

$f'(x) = -e^{-x} \sin x + e^{-x} \cos x = e^{-x}(\cos x - \sin x)$

$f''(x) = -e^{-x}(\cos x - \sin x) + e^{-x}(-\sin x - \cos x)$

$= -2e^{-x}(\cos x)$

Critical points are where $f'(x) = 0$ $\quad x = \frac{\pi}{4} + 2\pi n, \frac{5\pi}{4} + 2\pi n$

$f''(\frac{\pi}{4} + 2\pi n) = -2e^{-(\frac{\pi}{4} + 2\pi n)} \cdot \frac{\sqrt{2}}{2} < 0 \quad \frown$

$f''(\frac{5\pi}{4} + 2\pi n) = -2e^{-(\frac{5\pi}{4} + 2\pi n)} \cdot \frac{-\sqrt{2}}{2} > 0 \quad \smile$

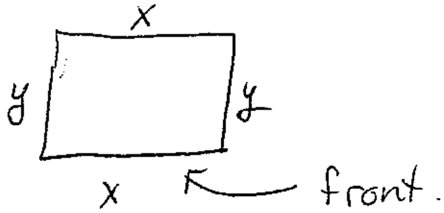
By 2^{nd} derivative test, local min at $x = \frac{5\pi}{4} + 2\pi n$, local max at $x = \pi/4 + 2\pi n$.

$- \quad 0 \quad + \quad 0 \quad - \quad 0 \quad + \quad 0$ $\quad f''$

$\frac{\pi}{2} \quad \frac{3\pi}{2} \quad \frac{5\pi}{2} \quad \frac{7\pi}{2}$

Inflection points occur at $x = \frac{\pi}{2} + 2\pi n, \frac{3\pi}{2} + 2\pi n$.

14.



Area $= 1200\ ft^2 = xy$

Cost $= 5x + 3x + 3y + 3y$ dollars

$= 8x + 6y$

Minimize $C = 8x + 6y$

$y = 1200/x$ so $C = 8x + \frac{7200}{x}$

also $0 \leq x < \infty$

$\frac{dC}{dx} = 8 - \frac{7200}{x^2} = 0$ when $x^2 = \frac{7200}{8} = 900$

$x = 30.$ $x = 0, -30$ are ~~is~~ also ~~a~~ critical points but are not physical possibilities.

First derivative test indicates a local (and absolute) minimum at $x = 30$

When $x = 30$, $y = \frac{1200}{30} = 40$

Answers: The dimensions are $x = 30$ ft., $y = 40$ ft.

and the minimum cost is

$C = 8(30) + 6(40) = \$480$

16. a) F b) T c) T d) F e) F f) F

g) T (Apply L'Hopital's rule enough times) h) F

i) F (∞^∞ is not indeterminate) j) F (0^∞ not indeterminate)

k) F l) T (you need to first get it in the form $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$). m) F (f must be continuous to ensure this) n) T o) F p) T q) T

r) T s) F (the concavity must change) t) T

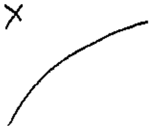
u) F (local min) v) F (could have local max or min - more information is needed) w) F

17. a) $f(x) = -(x-1)^2$ 

b) $f(x) = x(x-1)(x+1)$ 

c) $f(x) = e^x$ 

d) $f(x) = e^{-x}$ 

e) $f(x) = -e^{-x}$ 

f) $f(x) = -e^x$