

Test 3 Review Problems - Solutions.

1. Let $t =$ time in min., $r(t) =$ radius in in.,
 $V(t) =$ volume in in^3 .

Given: $V'(t) = 10$.

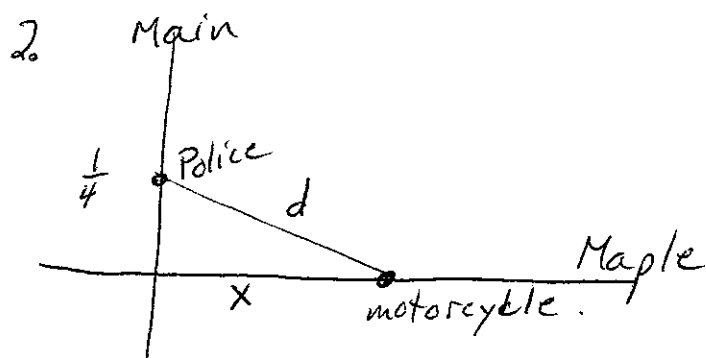
Find: $r'(t)$ when $r(t) = 5$.

$$V(t) = \frac{4}{3} \pi (r(t))^3.$$

$$V'(t) = 4\pi (r(t))^2 r'(t).$$

$$10 = 4\pi (5)^2 r'(t)$$

$$\boxed{r'(t) = \frac{\pi}{10} \text{ in/min.}}$$



Given $x'(t) = -40$

want $d'(t)$ when $x(t) = \frac{1}{8}$

$$\frac{1}{16} + (x(t))^2 = (d(t))^2$$

$$2(x(t))x'(t) = 2d(t)d'(t)$$

Let $x(t) =$ distance of motorcycle from intersection, in miles.

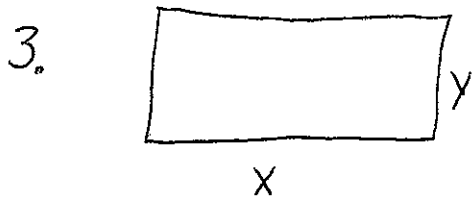
Let $d(t) =$ distance of motorcycle from police car, in miles.

Let $t =$ time in hours.

$$\text{when } x(t) = \frac{1}{8}, d(t) = \sqrt{\frac{1}{16} + \frac{1}{64}} = \sqrt{\frac{5}{64}} = \frac{\sqrt{5}}{8}$$

$$2 \cdot \frac{1}{8} (-40) = 2 \cdot \frac{\sqrt{5}}{8} d'(t).$$

$$\boxed{d'(t) = \frac{-40}{\sqrt{5}} \text{ miles/hr.}}$$



Let $t =$ time in sec.

$x(t) =$ base in cm

$y(t) =$ height in cm.

$A(t) =$ area in cm^2

Given

$$x'(t) = 4$$

$$y'(t) = -3$$

$$A(t) = x(t) y(t)$$

Want $A'(t)$ when

$$A'(t) = x'(t) y(t) + x(t) y'(t)$$

$$x = 20, y = 12.$$

$$A'(t) = 4 \cdot 12 + 20(-3)$$

$$= 48 - 60 = -12$$

The area is decreasing at $12 \text{ cm}^2/\text{sec}$.

4. a. $\int x^{1/2} dx = \frac{2}{3} x^{3/2} + C$ ← don't forget $+C$

b. $\int \csc^2 \theta d\theta = -\cot \theta + C$

c. $\int x^{-1} dx = \ln|x| + C$

d. $u = e^{-t}$ $du = -e^{-t} dt$ $-2 \int du = -2u + C = \boxed{-2e^{-t} + C}$

e. $u = 1 - x^2$
 $du = -2x dx$

$$\int x(1-x^2)^{1/3} dx = \frac{-1}{2} \int -2x(1-x^2)^{1/3} dx$$

$$= \frac{-1}{2} \int u^{1/3} du = \frac{-1}{2} \cdot \frac{3}{4} u^{4/3} + C$$

$$= \boxed{-\frac{3}{8} (1-x^2)^{4/3} + C}$$

$$f. \int \frac{1}{x^2+1} dx = \tan^{-1} x + C$$

$$g. \boxed{\begin{array}{l} u = x^2 + 1 \\ du = 2x dx \end{array}} \quad \int \frac{x}{x^2+1} dx = \frac{1}{2} \int \frac{2x}{x^2+1} dx$$
$$= \frac{1}{2} \int \frac{1}{u} du = \frac{1}{2} \ln|u| + C$$
$$= \frac{1}{2} \ln(x^2+1) + C$$

h. This one is a little tricky.

$$\boxed{\begin{array}{l} u = x^2 \\ du = 2x dx \end{array}} \quad \int \frac{x}{x^4+1} dx = \frac{1}{2} \int \frac{2x}{(x^2)^2+1} dx$$
$$= \frac{1}{2} \int \frac{1}{u^2+1} du = \frac{1}{2} \tan^{-1} u + C = \frac{1}{2} \tan^{-1}(x^2) + C$$

$$i. \int \frac{x^4+1}{\sqrt{x}} dx = \int \frac{x^4}{\sqrt{x}} + \frac{1}{\sqrt{x}} dx = \int x^{7/2} + x^{1/2} dx$$
$$= \frac{2}{9} x^{9/2} + \frac{2}{3} x^{3/2} + C$$

$$j. \int 3\cos x + 4\sin x dx = 3\sin x - 4\cos x + C$$

$$5. \sum_{i=2}^4 (2^i + 3i) = (2^2 + 3 \cdot 2) + (2^3 + 3 \cdot 3) + (2^4 + 3 \cdot 4)$$

$$= 4 + 6 + 8 + 9 + 16 + 12 = 55$$

$$6. \sum_{i=1}^{100} (4 + 3i - i^2) = \sum_{i=1}^{100} 4 + 3 \sum_{i=1}^{100} i - \sum_{i=1}^{100} i^2$$

$$= 4 \cdot 100 + 3 \cdot \frac{100(101)}{2} - \frac{100(101)(201)}{6}$$

$$\neq 13576, 90101 \quad = 338, 350$$

$$7. \sum_{i=1}^n \frac{1}{n} \left[\left(\frac{i}{n} \right)^2 - 5 \left(\frac{i}{n} \right) \right] = \frac{1}{n} \left[\frac{1}{n^2} \sum_{i=1}^n i^2 - \frac{5}{n} \sum_{i=1}^n i \right]$$

$$= \frac{1}{n} \left[\frac{1}{n^2} \frac{n(n+1)(2n+1)}{6} - \frac{5}{n} \cdot \frac{n(n+1)}{2} \right]$$

$$= \frac{2n^2 + 3n + 1}{6n^2} - \frac{5n + 5}{2n}$$

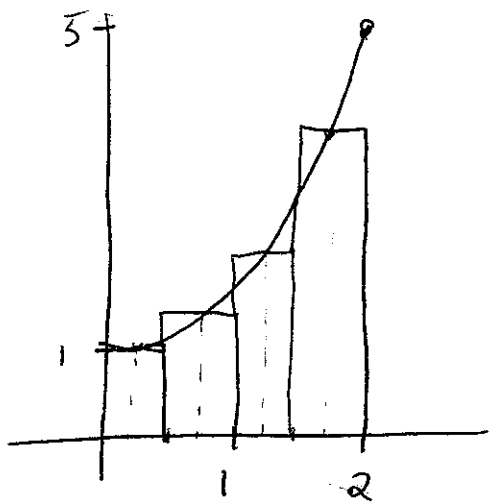
$$\lim_{n \rightarrow \infty} \left(\frac{2n^2 + 3n + 1}{6n^2} - \frac{5n + 5}{2n} \right) = \frac{1}{3} - \frac{5}{2} = \frac{2-15}{6} = \frac{-13}{6}$$

Bonus $\int_0^1 x^2 - 5x \, dx = \left. \frac{1}{3} x^3 - \frac{5}{2} x^2 \right|_0^1 = \frac{1}{3} - \frac{5}{2} = \frac{-13}{6}$

I chose $a=0$, $b=1$ so that

$x_i = \frac{i}{n}$, which appears in the sum, and
 $\Delta x = \frac{1}{n}$.

8. $a=0, b=2, n=4$



$$\Delta x = \frac{2-0}{4} = \frac{1}{2}$$

$$x_0=0, x_1=\frac{1}{2}, x_2=1, x_3=\frac{3}{2}, x_4=2$$

$$c_i = \text{midpoint of } [x_{i-1}, x_i] \\ = \frac{1}{2}(x_{i-1} + x_i)$$

$$\text{Riemann Sum} = \sum_{i=1}^4 f(c_i) \Delta x$$

$$= f\left(\frac{1}{4}\right) \cdot \frac{1}{2} + f\left(\frac{3}{4}\right) \cdot \frac{1}{2} + f\left(\frac{5}{4}\right) \cdot \frac{1}{2} + f\left(\frac{7}{4}\right) \cdot \frac{1}{2}$$

$$= \frac{1}{2} \left[\left(\frac{1}{16} + 1\right) + \left(\frac{9}{16} + 1\right) + \left(\frac{25}{16} + 1\right) + \left(\frac{49}{16} + 1\right) \right]$$

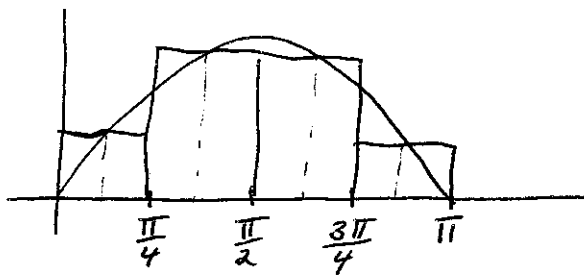
$$= \frac{1}{2} \left[\frac{84}{16} + 4 \right] = \frac{37}{8}$$

9. $f(x) = \sin x, a=0, b=\pi, n=4$

$$\Delta x = \frac{\pi-0}{4} = \frac{\pi}{4}$$

$$x_0=0, x_1=\frac{\pi}{4}, x_2=\frac{\pi}{2},$$

$$x_3=\frac{3\pi}{4}, x_4=\pi$$



$$c_i = \text{midpoint.}$$

$$c_1 = \frac{\pi}{8}, c_2 = \frac{3\pi}{8}, c_3 = \frac{5\pi}{8}, c_4 = \frac{7\pi}{8}$$

$$\text{Riemann sum} = \frac{\pi}{4} \left(\sin\left(\frac{\pi}{8}\right) + \sin\left(\frac{3\pi}{8}\right) + \sin\left(\frac{5\pi}{8}\right) + \sin\left(\frac{7\pi}{8}\right) \right)$$

$$\approx 2.05 \text{ (using calculator)}$$

$$10. \int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(c_i) \Delta x$$

where $\Delta x = \frac{b-a}{n}$

~~x_0, x_1, \dots, x_n~~ x_0, x_1, \dots, x_n is a partition of $[a, b]$ into n equal pieces.

c_i is a point in $[x_{i-1}, x_i]$.

11. $a=1, b=3, f(x) = x^2 - x$

$$\Delta x = \frac{2}{n}, \quad x_i = a + i \Delta x = 1 + \frac{2i}{n}$$

choose $c_i = x_i$

$$\sum_{i=1}^n f(c_i) \Delta x = \frac{2}{n} \sum_{i=1}^n f\left(1 + \frac{2i}{n}\right)$$

$$= \frac{2}{n} \sum_{i=1}^n \left[\left(1 + \frac{2i}{n}\right)^2 - \left(1 + \frac{2i}{n}\right) \right]$$

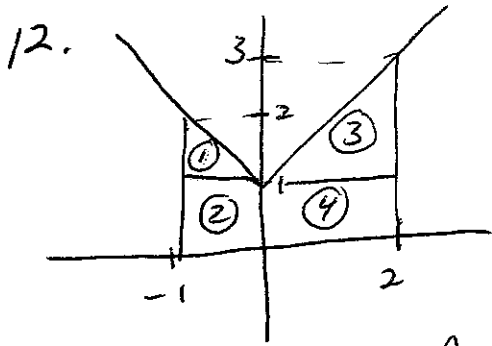
$$= \frac{2}{n} \sum_{i=1}^n \left(1 + \frac{4i}{n} + \frac{4i^2}{n^2} \right) - \left(1 + \frac{2i}{n} \right)$$

$$= \frac{2}{n} \sum_{i=1}^n \left[\frac{2i}{n} + \frac{4i^2}{n^2} \right]$$

$$= \frac{2}{n} \left[\frac{2}{n} \sum_{i=1}^n i + \frac{4}{n^2} \sum_{i=1}^n i^2 \right] = \frac{2}{n} \left[\frac{2}{n} \frac{n(n+1)}{2} + \frac{4}{n^2} \frac{n(n+1)(2n+1)}{6} \right]$$

$$= \frac{2n+1}{n} + \frac{4}{3} (2n^2 + 3n + 1)$$

$$\lim_{n \rightarrow \infty} \left[\frac{2n+1}{n} + \frac{2(2n^2+3n+1)}{3} \right] = 2 + \frac{8}{3} = \frac{14}{3}$$



Split the area into rectangles and triangles as shown.

$$\begin{aligned} \text{Area} &= \textcircled{1} + \textcircled{2} + \textcircled{3} + \textcircled{4} \\ &= \frac{1}{2} \cdot 1 \cdot 1 + 1 \cdot 1 + \frac{1}{2} \cdot 2 \cdot 2 + 2 \cdot 1 \\ &= \frac{1}{2} + 1 + 2 + 2 = 5\frac{1}{2} \end{aligned}$$

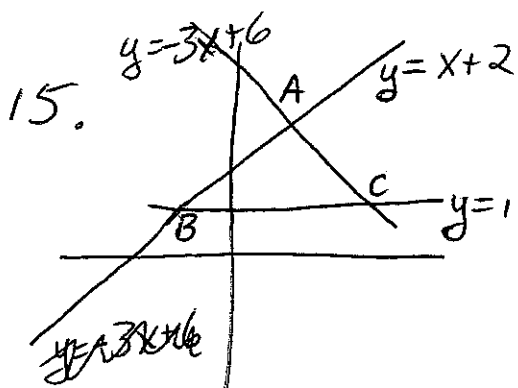
13. ave value = $\frac{1}{\pi/2} \int_0^{\pi/2} \cos(2x) dx$

$u = 2x \quad du = 2dx$ when $x=0, u=0$, when $x=\pi/2, u=\pi$

$$= \frac{2}{\pi} \cdot \frac{1}{2} \int_0^{\pi/2} \cos(2x) 2dx = \frac{1}{\pi} \int_0^{\pi} \cos u du$$

$$= \frac{1}{\pi} \sin u \Big|_0^{\pi} = \frac{1}{\pi} (\sin \pi - \sin 0) = \frac{1}{\pi} (0 - 0) = 0.$$

14. $f'(x) = (x3^x) 2x$ so $f'(2) = 72$



Find points of intersection

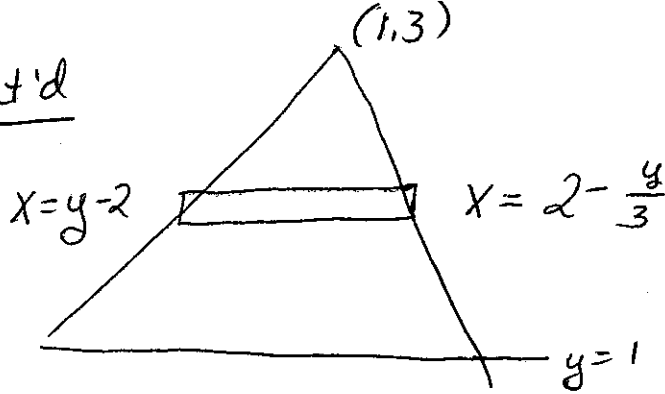
A. $x+2 = -3x+6 \quad 4x=4 \quad x=1$
 $y=3$

B. $x+2=1 \quad x=-1 \quad y=1$

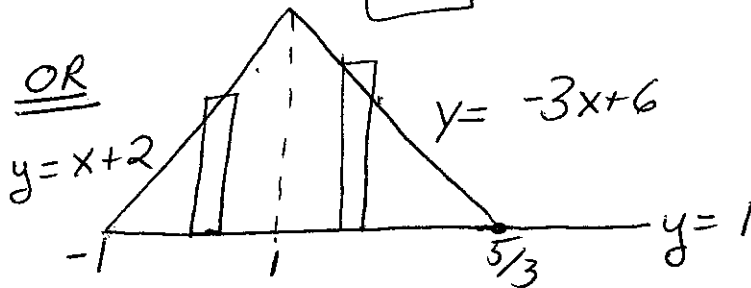
C. $-3x+6=1 \quad x=\frac{5}{3} \quad y=1$

→

15. cont'd

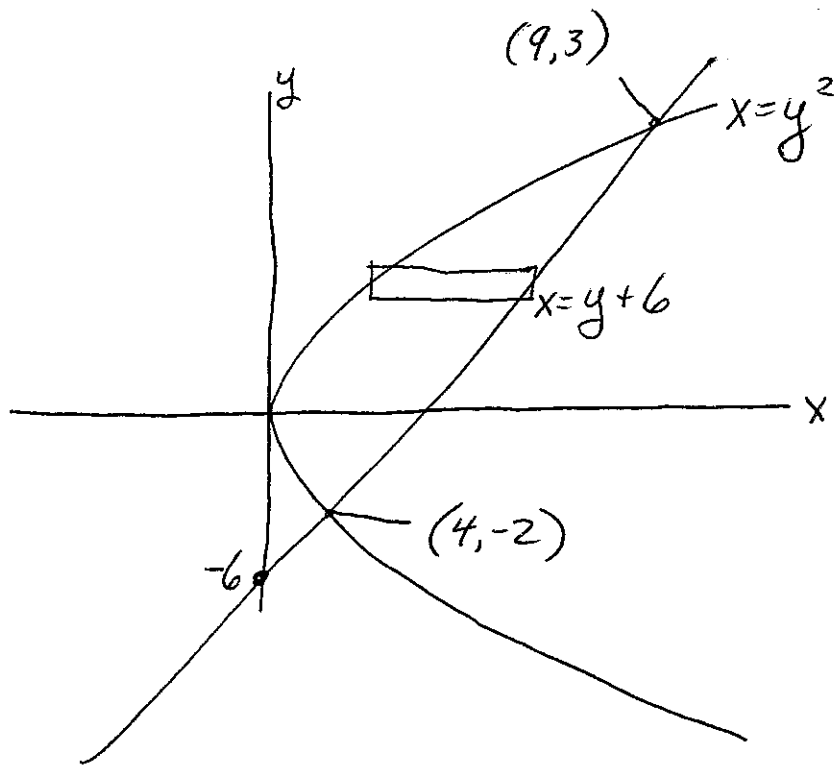


$$\begin{aligned} \text{area} &= \int_1^3 \left(2 - \frac{4}{3}\right) - (y-2) \, dy = \int_1^3 4 - \frac{4}{3}y \, dy \\ &= 4y - \frac{2}{3}y^2 \Big|_1^3 = 12 - 6 - \left(4 - \frac{2}{3}\right) \\ &= 2 + \frac{2}{3} = \boxed{\frac{8}{3}} \end{aligned}$$



$$\begin{aligned} \text{area} &= \int_{-1}^1 (x+2) \, dx + \int_1^{5/3} (-3x+6) \, dx \\ &= \int_{-1}^1 (x+1) \, dx + \int_1^{5/3} (-3x+5) \, dx \\ &= \left. \frac{1}{2}x^2 + x \right|_{-1}^1 + \left. \left(-\frac{3}{2}x^2 + 5x \right) \right|_1^{5/3} \\ &= \frac{1}{2} + 1 + \left(\frac{1}{2} - 1 \right) + \left(-\frac{3}{2} \left(\frac{5}{3} \right)^2 + 5 \left(\frac{5}{3} \right) \right) - \left(-\frac{3}{2} + 5 \right) \\ &= \boxed{\frac{8}{3}} \end{aligned}$$

16.



Find points of intersection $y^2 = y + 6$

$$y^2 - y - 6 = 0; \quad (y-3)(y+2) = 0; \quad y = 3, -2$$

when $y = 3$, $x = 9$, when $y = -2$, $x = 4$.

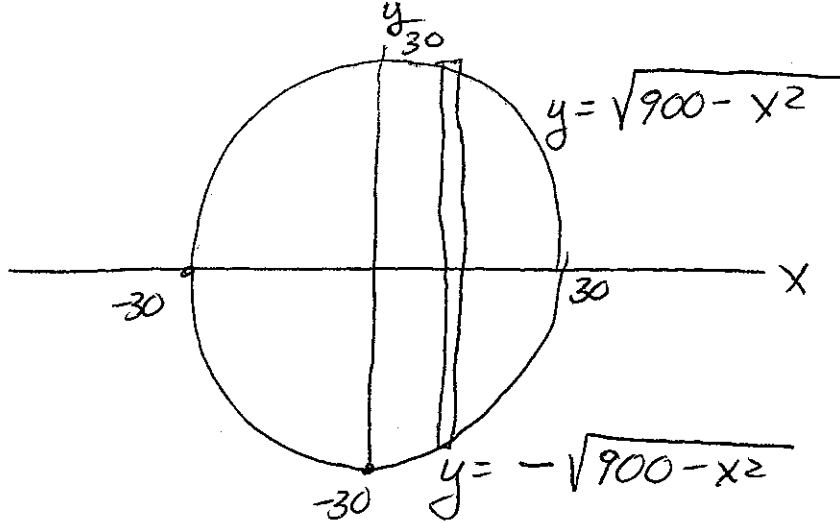
$$\text{area} = \int_{-2}^3 (y+6) - y^2 \, dy$$

$$= \left. \frac{1}{2}y^2 + 6y - \frac{1}{3}y^3 \right|_{-2}^3$$

$$= \frac{9}{2} + 18 - 9 - \left(2 - 12 + \frac{8}{3} \right)$$

$$= \frac{125}{6}$$

17.



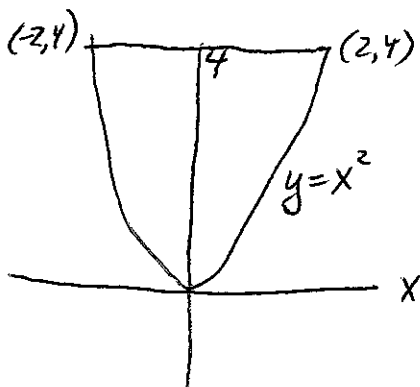
$$A(x) = \left(\sqrt{900-x^2} + \sqrt{900-x^2} \right)^2 = 4(900-x^2)$$

$$\text{Volume} = \int_{-30}^{30} 4(900-x^2) dx = 4 \left(900x - \frac{x^3}{3} \right) \Big|_{-30}^{30}$$

$$= 4 \left(27,000 - \frac{27,000}{3} - \left(-27,000 + \frac{27,000}{3} \right) \right)$$

$$= 4(54,000 - 18,000) = 4 \cdot 36,000 = 144,000 \text{ ft}^3$$

18. 5.2/26a



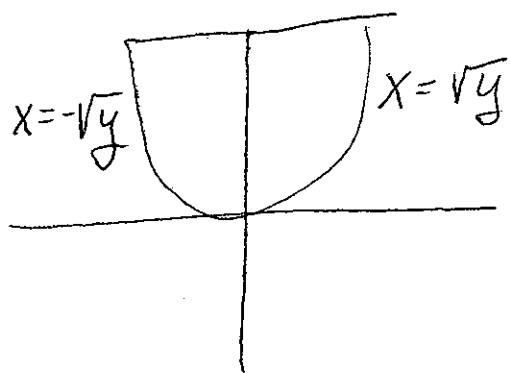
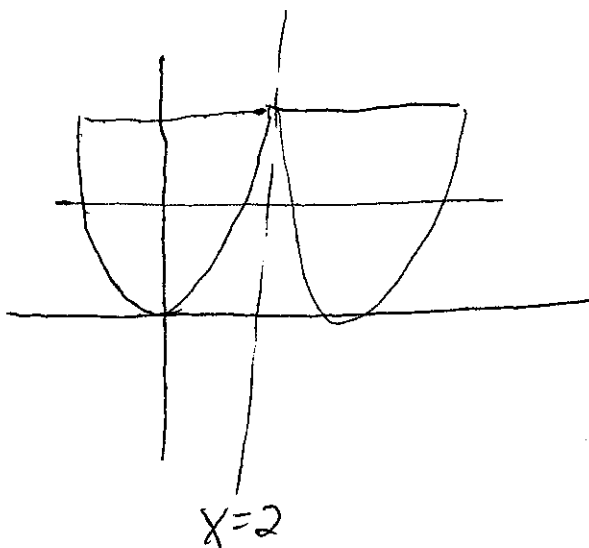
$$\text{Vol} = \int_{-2}^2 \pi (x^2)^2 dx$$

$$= \frac{\pi}{5} x^5 \Big|_{-2}^2 = \frac{\pi}{5} (32 - (-32))$$

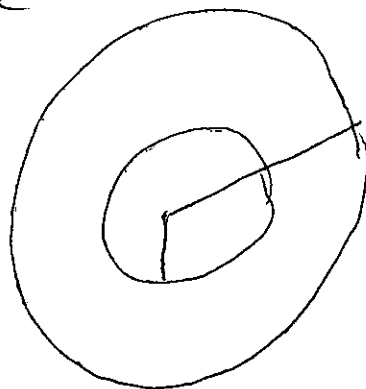
$$= \frac{64\pi}{5}$$

18. cont'd

(26e)



Slice



$$\text{outer radius} = 2 + \sqrt{y}$$

$$\text{inner radius} = 2 - \sqrt{y}$$

$$\text{Vol} = \int_0^4 \pi (2 + \sqrt{y})^2 - \pi (2 - \sqrt{y})^2 dy$$

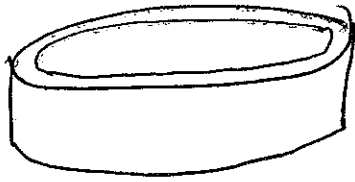
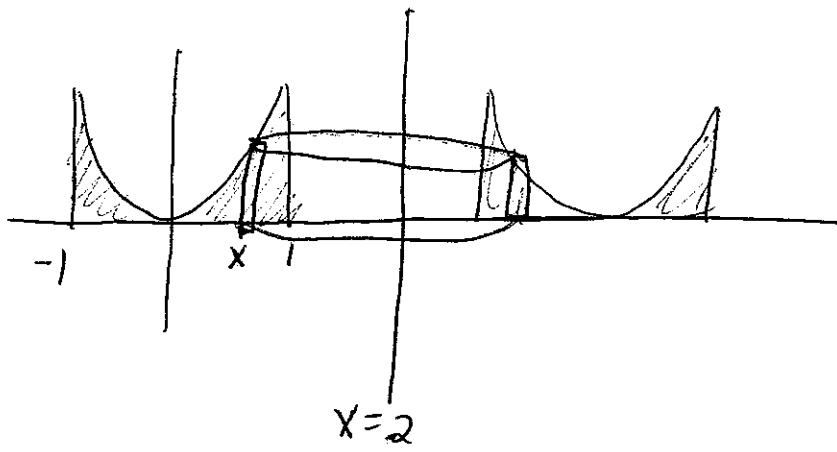
$$= \pi \int_0^4 (4 + 4y^{1/2} + y - (4 - 4y^{1/2} + y)) dy$$

$$= \pi \int_0^4 8y^{1/2} dy = 8\pi \frac{2}{3} y^{3/2} \Big|_0^4$$

$$= \frac{16\pi}{3} (4^{3/2} - 0^{3/2}) = \frac{16\pi}{3} \cdot 8 = \frac{128\pi}{3}$$

$\frac{16}{12} \cdot 8$

19.



height = $y = x^2$
radius = $2 - x$

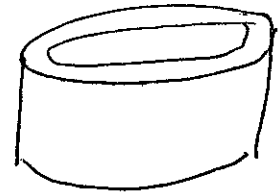
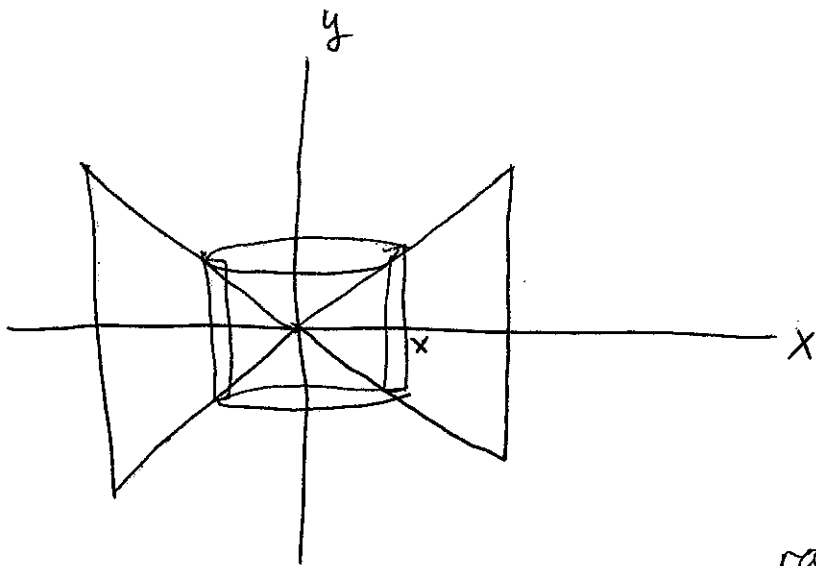
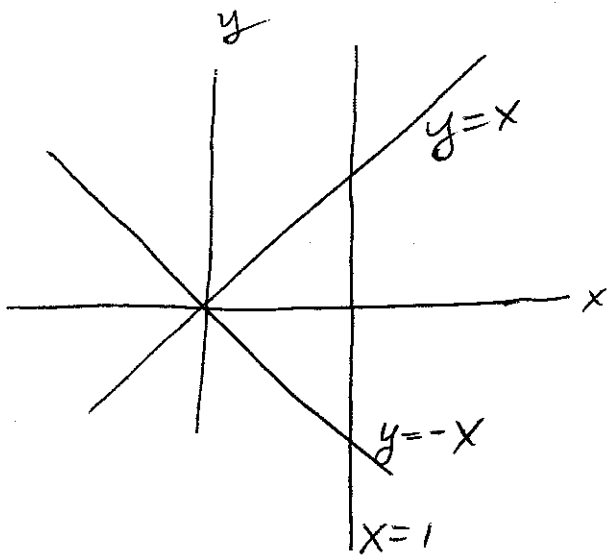
$$\text{Vol.} = \int_{-1}^1 2\pi(2-x)x^2 dx$$

$$= 2\pi \int_{-1}^1 2x^2 - x^3 dx = 2\pi \left(\frac{2}{3}x^3 - \frac{1}{4}x^4 \right) \Big|_{-1}^1$$

$$= 2\pi \left(\frac{2}{3} - \frac{1}{4} - \left(-\frac{2}{3} - \frac{1}{4} \right) \right)$$

$$= 2\pi \cdot \frac{4}{3} = \frac{8\pi}{3}$$

20.



$$\text{height} = y_2 - y_1 \\ = x - (-x) = 2x$$

$$\text{radius} = x$$

$$\text{Vol} = \int_0^1 2\pi x (2x) dx$$

$$= 4\pi \int_0^1 x^2 dx = \frac{4\pi}{3} x^3 \Big|_0^1 = \frac{4\pi}{3}$$

21. a. F

b. T $F(x) = \int_a^x f(t) dt$ FTC, Part II

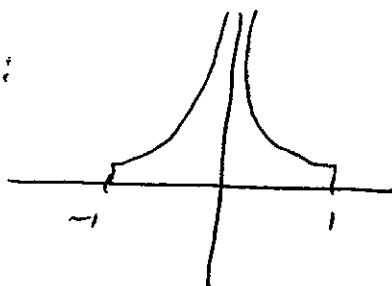
c. T (think of signed area - there is no area below the x-axis)

21 cont'd

d. F. FTC Part I applies only if the function is continuous.

Look at the graph:

If ~~is~~ $\int_{-1}^1 \frac{1}{x^2} dx$



is defined at all, it would certainly be positive, not -2 . In fact it is $+\infty$, as you will learn in Calc II.

e. T

f. T. See the Integral Mean Value Theorem.

g. F. ~~$f(x)$ is integrable~~ $f(x) = |x|$ is cont., therefore integrable, but it is not diff. at $x=0$.

h. T ~~is~~ by integral properties, Section 4.4

i. T See Section 4.4

j. T

k. F - you cannot factor an x past \int sign.

l. F - this just doesn't work. Try any example to see this.

22. See Section 4.5