Section 10.4 Summary

Taylor Polynomial

Given a function $f(x)$, a point $x=a$, and a natural number n,

Definition

the Taylor polynomial of degree n, centered at $x=a$, is

$$\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k =$$

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Properties

$$f(a) = P_n(a)$$
$$f'(a) = P_n'(a)$$
$$\vdots$$
$$f^{(n)}(a) = P_n^{(n)}(a)$$

The derivatives of f and P_n at a are the same, up to the n^{th} derivative

Remainder

How close is $P_n(x)$ to $f(x)$ for a given n, x?

Let $R_n(x) = f(x) - P_n(x)$ (remainder)

Taylor's formula says that

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1}$$ for some z between x and a (it doesn't say exactly which z)

Definition

<u>Taylor Series</u> Given a function f and a point $x=a$, the Taylor Series for f, centered at a, is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k .$$

Note this is an infinite series, not a (finite) polynomial.

<u>Maclaurin Series</u> When $a=0$, the Taylor series can also be called "Maclaurin Series".

Convergence

The partial sums for the Taylor Series are the Taylor Polynomials $P_n(x)$. For any fixed x, $\lim_{n\to\infty} P_n(x) = f(x)$ if and only if $\boxed{\lim_{n\to\infty} R_n(x) = 0}$.

So the Taylor Series converges (to $f(x)$) if .

Commonly occurring Examples

The following are examples of Maclaurin series which converge for all x. You should know how to derive these and how to show they converge. Memorizing them is recommended.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$