242 Homework #12 Solutions

13.6/3

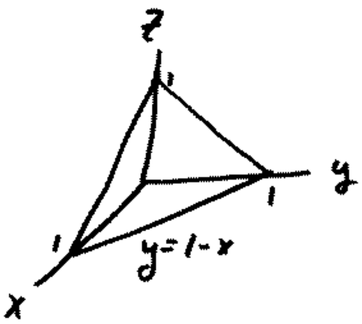
$$\int_{x=-1}^{x=3}\int_{y=0}^{y=2}\int_{z=-2}^{z=6} xyz \, dz\,dy\,dx$$

$$= \int_{x=-1}^{x=3}\int_{y=0}^{y=2} \tfrac{1}{2}xyz^2\Big|_{z=-2}^{z=6} \, dy\,dx$$

$$= \int_{x=-1}^{x=3}\int_{y=0}^{y=2} 16xy \, dy\,dx = \int_{x=1}^{x=3} 8xy^2\Big|_{y=0}^{y=2} \, dx$$

$$= \int_1^3 32x\,dx = 16x^2\Big|_1^3 = 16(9-1) = 128$$

13.6/5



$0 \le z \le 1-x-y$
$0 \le y \le 1-x$
$0 \le x \le 1$

$$\int_{x=0}^{x=1}\int_{y=0}^{y=1-x}\int_{z=0}^{z=1-x-y} x^2 \, dz\,dy\,dx$$

$$= \int_{x=0}^{x=1}\int_{y=0}^{y=1-x} x^2(1-x-y)\,dy\,dx$$

→

13.6/5 cont'd

$$= \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} x^2 - x^3 - x^2 y \; dy\, dx$$

$$= \int_{x=0}^{x=1} x^2 y - x^3 y - \tfrac{1}{2} x^2 y^2 \Big|_{y=0}^{y=1-x} dx$$

$$= \int_{x=0}^{x=1} x^2(1-x) - x^3(1-x) - \tfrac{1}{2} x^2 (1 - 2x + x^2) \; dx$$

$$= \int_0^1 x^2 - \cancel{x^3} - x^3 + x^4 - \tfrac{1}{2} x^2 + \cancel{x^3} - \tfrac{1}{2} x^4 \; dx$$

$$= \int_0^1 \tfrac{1}{2} x^4 - x^3 + \tfrac{1}{2} x^2 \, dx = \tfrac{1}{10} x^5 - \tfrac{1}{4} x^4 + \tfrac{1}{6} x^3 \Big|_0^1$$

$$= \frac{1}{10} - \frac{1}{4} + \frac{1}{6} = \frac{6 - 15 + 10}{60} = \frac{1}{60}$$

13.6/9 Intersection of surfaces: ~~$y^2 = 8 - y^2$;~~

~~$2y^2 = 8 \quad y^2 = 4 \quad y = \pm 2$~~ $\quad 2 - x^2 = x^2$; $2 = 2x^2$; $x = \pm 1$

$2 - x^2 \le z \le x^2 \qquad -1 \le x \le 1, \qquad 0 \le y \le 3$

should be switched, so I put a -sign in front.

$$-\int_{x=-1}^{x=1} \int_{y=0}^{y=3} \int_{z=2-x^2}^{z=x^2} x + y \; dz\, dy\, dx \quad \rightarrow$$

13.6/9 cont'd

$$= -\int_{x=-1}^{x=1} \int_{y=0}^{y=3} xz + yz \Big|_{z=2-x^2}^{z=x^2} \, dy\, dx$$

$$= -\int_{x=-1}^{x=1} \int_{y=0}^{y=3} x^3 + yx^2 - [2x - x^3 + 2y - yx^2] \, dy\, dx$$

$$= -\int_{x=-1}^{x=1} \int_{y=0}^{y=3} 2x^3 + 2yx^2 - 2x - 2y \quad dy\, dx$$

$$= -\int_{x=-1}^{x=1} 2x^3y + y^2x^2 - 2xy - y^2 \Big|_{y=0}^{y=3} dx$$

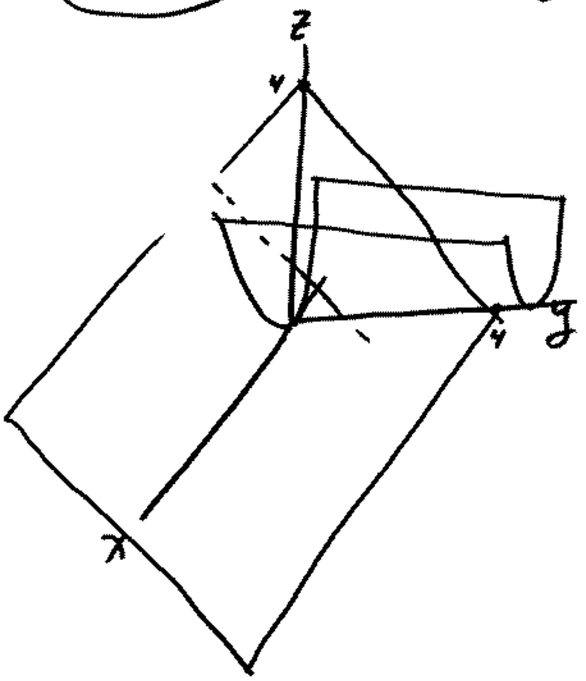
$$= -\int_{x=-1}^{x=1} 6x^3 + 9x^2 - 6x - 9 \; dx$$

$$= -\left[\tfrac{3}{2}x^4 + 3x^3 - 3x^2 - 9x \Big|_{-1}^{1}\right]$$

$$= -\left[\tfrac{3}{2} + 3 - 3 - 9 - \left(\tfrac{3}{2} - 3 - 3 + 9\right)\right]$$

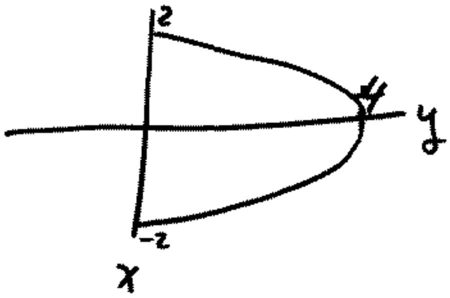
$$= -\left[\cancel{\tfrac{3}{2}} + \cancel{3} - \cancel{3} - 9 - \cancel{\tfrac{3}{2}} + 3 + 3 - 9\right] = -[6 - 18] = +12$$

13.6/17 $\quad z = x^2 \quad y + z = 4 \quad y = 0 \quad z = 0$



Find intersection of $z = x^2$ and $y + z = 4$ $\quad (z = 4 - y)$

$x^2 = 4 - y \qquad y = 4 - x^2$



$-2 \le x \le 2$

$0 \le y \le 4 - x^2$

$x^2 \le z \le 4 - y$

$$\int_{x=-2}^{x=2} \int_{y=0}^{y=4-x^2} \int_{z=x^2}^{z=4-y} 1 \, dz \, dy \, dx = \text{Volume}$$

$$= \int_{x=-2}^{x=2} \int_{y=0}^{y=4-x^2} 4 - y - x^2 \, dy \, dx$$

$$= \int_{x=-2}^{x=2} 4y - \tfrac{1}{2}y^2 - x^2 y \Big|_{y=0}^{y=4-x^2} dx$$

$$= \int_{x=-2}^{x=2} 4(4-x^2) - \tfrac{1}{2}(4-x^2)^2 - x^2(4-x^2) \, dx$$

$\rightarrow$

13.6/17 cont'd

$$= \int_{x=-2}^{x=2} 16-4x^2-\left(8-4x^2+\tfrac{1}{2}x^4\right)-4x^2+x^4 \, dx$$

$$= \int_{x=-2}^{x=2} \tfrac{1}{2}x^4-4x^2+8 \, dx = \tfrac{1}{10}x^5-\tfrac{4}{3}x^3+8x \Big|_{-2}^{2}$$

$$= \frac{32}{10} - \frac{32}{3}+16-\left(-\frac{32}{10}+\frac{32}{3}-16\right)$$

$$= 2\left(\frac{16}{5}-\frac{32}{3}+16\right) = 2\cdot\frac{48-160+240}{15} = 2\cdot\frac{128}{15} = \frac{256}{15}$$

13.6/23

$$\bar{x} = \int_{x=-2}^{x=2}\int_{y=0}^{y=4-x^2}\int_{z=x^2}^{z=4-y} x \, dz\, dy\, dx$$

(x just carries along until 3rd integration)

$$= \int_{x=-2}^{x=2} \tfrac{1}{2}x^5-4x^3+8x \, dx = \tfrac{1}{12}x^6-x^4+4x^2\Big|_{-2}^{2} = 0$$

$$\bar{y} = \int_{x=-2}^{x=2}\int_{y=0}^{y=4-x^2}\int_{z=x^2}^{z=4-y} y \, dz\, dy\, d\cancel{x}$$

$$= \int_{x=-2}^{x=2}\int_{y=0}^{y=4-x^2} 4y-y^2-x^2y \, dy\, d\cancel{x} \quad \rightarrow$$

13.6/36

$$\text{mass} = m = \int_{x=0}^{x=1}\int_{y=0}^{y=1}\int_{z=0}^{z=1} kz\, dz\, dy\, dx$$

$$= \int_{x=0}^{x=1}\int_{y=0}^{y=1} \frac{k}{2}z^2\Big|_0^1\, dy\, dx$$

$$= \int_{x=0}^{x=1}\int_{y=0}^{y=1} \frac{k}{2}\, dy\, dx = \int_{x=0}^{x=1}\frac{k}{2}\, dx = k/2$$

$$\bar{x} = \frac{2}{k}\int_{x=0}^{x=1}\int_{y=0}^{y=1}\int_{z=0}^{z=1} k\, xz\, dz\, dy\, dx$$

$$= 2\int_{x=0}^{x=1}\int_{y=0}^{y=1} x\cdot\frac{1}{2}z^2\Big|_{z=0}^{z=1}\, dy\, dx = \int_{x=0}^{x=1}\int_{y=0}^{y=1} x\, dy\, dx$$

$$= \int_{x=0}^{x=1} x\, dx = \frac{1}{2}x^2\Big|_0^1 = \boxed{\frac{1}{2}}.$$

$$\bar{y} = \frac{2}{k}\int_{x=0}^{x=1}\int_{y=0}^{y=1}\int_{z=0}^{z=1} k\, yz\, dz\, dy\, dx$$

$$= 2\int_{x=0}^{x=1}\int_{y=0}^{y=1} y\cdot\frac{1}{2}z^2\Big|_{z=0}^{z=1}\, dy\, dx = \int_{x=0}^{x=1}\int_{y=0}^{y=1} y\, dy\, dx$$

$$= \int_{x=0}^{x=1}\frac{1}{2}y^2\Big|_0^1\, dx = \int_0^1 \frac{1}{2}\, dx = \boxed{1/2}$$

$\rightarrow$

13.6/36 cont'd

$$\bar{z} = \frac{2}{k}\int_{x=0}^{x=1}\int_{y=0}^{y=1}\int_{z=0}^{z=1} kz^2\,dz\,dy\,dx$$

$$= 2\int_{x=0}^{x=1}\int_{y=0}^{y=1} \frac{1}{3}z^3\Big|_0^1\,dy\,dx = \frac{2}{3}\int_{x=0}^{x=1}\int_{y=0}^{y=1} 1\,dy\,dx = \boxed{\frac{2}{3}}$$

centroid $\boxed{(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{1}{2}, \frac{1}{2}, \frac{2}{3}\right)}$

13.6/37

$$I_z = \int_{x=0}^{x=1}\int_{y=0}^{y=1}\int_{z=0}^{z=1} (x^2+y^2)kz\,dz\,dy\,dx$$

$$= k\int_{x=0}^{x=1}\int_{y=0}^{y=1} (x^2+y^2)\frac{1}{2}z^2\Big|_0^1\,dy\,dx$$

$$= \frac{k}{2}\int_{x=0}^{x=1}\int_{y=0}^{y=1} x^2+y^2\,dy\,dx = \frac{k}{2}\int_{x=0}^{x=1} x^2y+\frac{1}{3}y^3\Big|_{y=0}^{y=1}\,dx$$

$$= \frac{k}{2}\int_{x=0}^{x=1} x^2+\frac{1}{3}\,dx = \frac{k}{2}\left(\frac{1}{3}x^3+\frac{1}{3}x\right)\Big|_0^1 = \frac{k}{2}\cdot\frac{2}{3} = \boxed{\frac{k}{3}}$$