

**MATH242 PRACTICE PROBLEMS FOR THE FINAL
EXAM**

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- (1) Compute $\frac{\partial f}{\partial x} = f_x$, $\frac{\partial f}{\partial y} = f_y$, $\frac{\partial^2 f}{\partial x^2} = f_{xx}$, $\frac{\partial^2 f}{\partial y^2} = f_{yy}$, $\frac{\partial^2 f}{\partial xy} = f_{xy}$, $\frac{\partial^2 f}{\partial zy} = f_{zy}$, $\frac{\partial^3 f}{\partial xyz} = f_{xyz}$ for the following functions;

Extra part: Additionally compute the laplacian Δf for all functions f given above.

Note that the laplacian of a given function f is defined by $\Delta f := \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = f_{xx} + f_{yy} + f_{zz}$.

- (a) $f(x, y, z) = e^{xyz}$,
- (b) $f(x, y, z) = e^{(x^2+y^3)z}$,
- (c) $f(x, y, z) = \cos(x^2 + y) \arctan(z^3 + 2x)$,
- (d) $f(x, y, z) = e^{xy+z^2} (\cos(xy + y^2z) + \sin(x^2y + z^3))$.

- (2) A function u is called harmonic in case it satisfies Laplace's equation $\Delta u = \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$ which can also be written as $u_{xx} + u_{yy} + u_{zz} = 0$. Show that all of the functions listed below are harmonic

- (a) $u = x^2 - y^2 + z$,
- (b) $u = \ln \sqrt{x^2 + y^2}$,
- (c) $u = \arctan(\frac{y}{x})$,
- (d) $u = \arctan(\frac{x}{y})$,
- (e) $u = e^{-x} e^{-y} \text{Cos}(\sqrt{2}z)$.

- (3) Let $w = f(x, y, z)$,

- (a) $x = \rho \cos(\theta)$, $y = \rho \sin(\theta)$, $z = z$. Write w_ρ , w_θ , w_z , $w_{\rho\rho}$, $w_{\theta\theta}$, w_{zz} .

Date: November 21, 2005.

- (b) Within the same settings above, compute $w_x, w_y, w_z, w_{xx}, w_{yy}, w_{zz}$, and hence the laplacian $\Delta w = \nabla^2 w = w_{xx} + w_{yy} + w_{zz}$ in terms of the cylindrical coordinates.
- (c) $x = \rho \sin(\phi) \cos(\theta), y = \rho \sin(\phi) \sin(\theta), z = \rho \cos(\phi)$. Write $w_\rho, w_\theta, w_\phi, w_{\rho\rho}, w_{\phi\phi}, w_{\theta\theta}$.
- (d) Within the same settings above, compute $w_x, w_y, w_z, w_{xx}, w_{yy}, w_{zz}$, and hence the laplacian $\Delta w = \nabla^2 w = w_{xx} + w_{yy} + w_{zz}$ in terms of the spherical coordinates.
- (4) You are standing at the point where $x = y = 100 \text{ ft}$. on a hillside whose height is given by $z = 100 + \frac{1}{100}(x^2 - 3xy + 2y^2)$ with the positive x -axis to the east, and the positive y -axis to the north.
- (a) If you head east, will you initially be ascending or descending?
- (b) If you head north, will you initially be ascending or descending?

Hint: Consider the directional derivatives z_x , and z_y of the height function z .

- (5) Let $F(x, y, z) = x^4 + y^4 + z^4 + 4x^2y^2z^2 - 34 = 0$.
- (a) Show that at $P(1, 1, 2)$ we can write $x = f(y, z)$. Find $\frac{\partial x}{\partial y}$.
- (b) Show that at $P(1, 2, 1)$ we can write $y = g(x, z)$. Find $\frac{\partial y}{\partial z}$.
- (c) Show that at $P(2, 1, 1)$ we can write $z = h(x, y)$. Find $\frac{\partial z}{\partial x}$.
- (6) $r = \frac{p}{q} + \frac{q}{s} + \frac{s}{p}, p = e^{yz}$ and $s = e^{xy}$. Find $\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}$, and $\frac{\partial r}{\partial z}$.
- (7) $x^3 + y^3 + z^3 = xyz$, assuming $z = f(x, y)$ find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$, and $\frac{\partial^2 z}{\partial y \partial z}$.
- (8) Let $w = uv - xy, u = \frac{x}{x^2+y^2}$, and $v = \frac{y}{x^2+y^2}$. Find w_x , and w_y .
- (9) Let $w = f(u)$, where $u = \frac{x^2-y^2}{x^2+y^2}$. Show that $xw_x + yw_y = 0$.

- (10) Find the derivative of the functions below at the given direction and also find the direction at which the directional derivative is maximum, and minimum
- (a) $f(x, y, z) = \sqrt{xyz + x^2y + xy^3 + z^4}$, $P(2, 1, 1)$, $v = \langle 1, 2, -2 \rangle$,
- (b) $f(x, y, z) = \ln(1+x^2+y^2-2z^2)$, $P(1, 1, -1)$, $v = \langle 2, -2, -3 \rangle$,
- (c) $f(x, y, z) = \arctan\left(\frac{x^2-y^2-z^2}{x^2+y^2+z^2}\right)$, $P\left(\frac{\pi}{4}, \frac{\pi}{4}, 0\right)$, $v = \langle 1, 1, -1 \rangle$.
- (11) Find the tangent line or plane at the given points
- (a) $2x^2 + 3y^2 = 35$, $P(2, 3)$,
- (b) $xyz + x^2 - y^2 + z^3 = 14$, $P(5, -2, 3)$,
- (c) $z^2 - \arctan\left(\frac{x^2-y^2}{x^2+y^2}\right) = 0$, $P\left(\frac{\pi}{4}, \frac{\pi}{4}, 0\right)$.
- (12) Find the equation for the plane tangent to the paraboloid $z = 2x^2 + 3y^2$, and simultaneously parallel to the plane $4x - 3y - z = 10$.
- (13) You are standing at the point $(-100, -100, 430)$ on a hill that has the shape of the graph $z = 500 - (0,003) \cdot x^2 - (0,004) \cdot y^2$ with the positive x -axis to the east, and the positive y -axis to the north.
- (a) If you head northwest will you initially be ascending or descending?
- (b) If you head northeast will you initially be ascending or descending?
- (c) In what direction should you proceed in order to climb most steeply?

Hint: Consider the gradient $\nabla z = \langle \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y} \rangle$ of the shape function z .

(14) By applying the method of Lagrange's multipliers, find the maximum and minimum values (if exist) of the functions with given constraints.

(a) $f(x, y) = x + y, x^2 + 4y^2 = 1,$

(b) $f(x, y) = x^2 + y^2, 2x + 3y = 6,$

(c) $f(x, y) = 4x^2 + 9y^2, x^2 + y^2 = 1,$

(d) $f(x, y, z) = 3x + 2y + z, x^2 + y^2 + z^2 = 1,$

(e) $f(x, y, z) = xyz, x^2 + y^2 + z^2 = 1,$

(f) $A = f(x, y, z, \alpha) = \frac{1}{2}xy \sin(\alpha)$, subject to the constraints $x + y + z = P$, a constant, and $z^2 = x^2 + y^2 - 2xy \cos(\alpha)$, the law of Cosines. (**Translation into Words:** What is the maximum/minimum area of a triangle, with side lengths x, y, z , and the constant perimeter P ?)

(g) A triangle with sides x, y , and z has fixed perimeter $2s = x + y + z$. Its area A is given by *Heron's formula*; $A = \sqrt{s(s-a)(s-b)(s-c)}$. Use the method of Lagrange's multipliers to show that, among all triangles with the given perimeter, the one with the largest area is equilateral. s. (

Hint: Since the function $\sqrt{\cdot}$ is strictly monotonic, consider maximizing A^2 instead of A .)

(h) (i) Suppose that x_1, x_2, \dots, x_n are positive. Show that the minimum value of $f(x_1, x_2, \dots, x_n) = x_1 + x_2 + \dots + x_n$ subject to the constraint $x_1 \cdot x_2 \cdot \dots \cdot x_n = 1$ is n .

(ii) Given n positive numbers a_1, a_2, \dots, a_n , let

$$x_i = \frac{a_i}{(a_1 \cdot a_2 \cdot \dots \cdot a_n)^{\frac{1}{n}}} \quad \text{for } i = 1, \dots, n.$$

and apply the result in (i) to deduce the arithmetic-geometric mean inequality

$$\sqrt[n]{a_1 \cdot a_2 \cdot \dots \cdot a_n} \leq \frac{a_1 + a_2 + \dots + a_n}{n}.$$

- (j) An ice tray, in the shape of a rectangular prism, is to be made from a material that costs $1\text{¢}/\text{in}^2$. The ice tray designed is supposed to have 12 compartments, each having a square horizontal cross section, and a total volume of 12 in^3 , ignoring the partitions. Considering the length, width, and the height of the tray to be x, y , and z units, the cost function is given by $f(x, y, z) = xy + 3xz + 7yz$. What is the minimum cost of production for this design of an ice tray?
- (15) By writing down the function to be extremized, the function/functions giving the constraint/constraints, and the auxiliary function generated while applying the method of Lagrange's multipliers step by step, solve the following extremal problems
- Find the points of the parabola $y = (x-1)^2$ that are closest to the origin,
 - Find the points of the ellipse $4x^2 + 9y^2 = 36$ that are closest to the origin,
 - Find the points of the surface $xyz = 1$ that is closest to the point $P(1, 2, 3)$,
 - Find the points on the sphere with center $(1, 2, 3)$ and radius 6 that are closest to and furthest from the origin,
 - Find the points of the ellipsoid $4x^2 + 9y^2 + z^2 = 36$ that are closest and furthest from the origin.
- (16) Find and classify the critical points of the functions listed below, by applying the Second Derivative test for a function of two variables.
- $f(x, y) = x^3 + y^3 + 3xy + 3$,
 - $f(x, y) = x^4 + y^4 - 4xy$,

- (c) $f(x, y) = (x^2 + y^2)e^{-x^2 - y^2}$,
- (d) $f(x, y) = x^4 + y^4$, **Hint:** $\Delta = f_{xx}f_{yy} - (f_{xy})^2$ is zero at the origin. Hence classify this critical point by visualizing the surface $z = f(x, y)$.
- (e) $f(x, y) = e^{-x^4 - y^4}$, **Hint:** $\Delta = f_{xx}f_{yy} - (f_{xy})^2$ is zero at the origin. Hence classify this critical point by visualizing the surface $z = f(x, y)$.
- (f) $f(x, y) = (x^2 + 2y^2)e^{\frac{1}{8}(x^2 + 4y^2)}$,
- (g) $f(x, y) = xy e^{\frac{1}{8}(x^2 + 4y^2)}$,
- (h) $f(x, y) = \sin\left(\frac{\pi x}{2}\right) \sin\left(\frac{\pi y}{2}\right)$,
- (i) $f(x, y) = \frac{xy(x^2 - y^2)}{x^2 + y^2}$.
- (j) $f(x, y) = x^3 - 3xy^2$, **Hint:** $\Delta = f_{xx}f_{yy} - (f_{xy})^2$ is zero at the origin. Hence classify this critical point by examining the behaviour of $x^3 - 3xy^2$ on straight lines through the origin.

(17) Describe the region on which the following iterated integrals are computed. Then reverse the order of integration, and at last evaluate the resulting integral.

(a)

$$\int_0^2 \int_{-\sqrt{2y}}^{\sqrt{2y}} (3x + 2y) dx dy$$

(b)

$$\int_0^2 \int_{\frac{y}{2}}^1 (x + y) dx dy$$

(c)

$$\int_0^1 \int_{\arctan y}^{\frac{\pi}{4}} \sec x dx dy$$

(d)

$$\int_{-2}^2 \int_{y^2 - 4}^{4 - y^2} y dx dy$$

(e)

$$\int_0^3 \int_0^y \sqrt{y^2 + 16} dx dy$$

(f)
$$\int_1^2 \int_0^{\sqrt{2x-x^2}} \frac{1}{\sqrt{x^2+y^2}} dy dx$$

(g)
$$\int_0^\infty \int_0^\infty \frac{1}{(1+x^2+y^2)^2} dx dy$$

(h)
$$\int_0^1 \int_y^1 \frac{\sin x}{x} dx dy$$

(i)
$$\int_0^\infty \int_0^\infty \frac{e^{-y}}{y} dy dx$$

(ii)
$$\int_0^1 \int_x^1 e^{-y^2} dy dx$$

(18) Evaluate the following double integrals by first converting to polar coordinates.

(a)
$$\int_1^2 \int_0^{\sqrt{2x-x^2}} \frac{1}{\sqrt{x^2+y^2}} dy dx$$

(b)
$$\int_0^\infty \int_0^\infty \frac{1}{(1+x^2+y^2)^2} dx dy$$

(c)
$$\int_0^1 \int_0^{\sqrt{1-x^2}} \frac{1}{\sqrt{4-x^2-y^2}} dy dx$$

(d)
$$\int_0^1 \int_0^{\sqrt{1-y^2}} \sin(x^2+y^2) dy dx$$

(e)
$$\iint_R \frac{1}{(1+x^2+y^2)^2} dA,$$

where R is the region in the first-quadrant, bounded by the curve $x^4 + x^2y^2 = y^2$ and the line $y = x$.

- (19) Let R be the first-quadrant region bounded by the lemniscates $r^2 = 3 \cos 2\theta$, $r^2 = 4 \cos 2\theta$ and $r^2 = 3 \sin 2\theta$, $r^2 = 4 \sin 2\theta$. Show that its area is

$$A = \frac{2\sqrt{17} - 5\sqrt{2}}{4}.$$

- (20) Substitute $u = x - y$ and $v = x + y$ to evaluate

$$\iint_R e^{\frac{x-y}{x+y}} dx dy,$$

where R is bounded by the coordinate axes and the line $x + y = 1$.

- (21) Change to spherical coordinates to show that, for $k > 0$,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sqrt{x^2 + y^2 + z^2} e^{-k(x^2+y^2+z^2)} dx dy dz = \frac{2\pi}{k^2}.$$

- (22) Let R be the first-quadrant region bounded by the circles $x^2 + y^2 = 2x$, $x^2 + y^2 = 6x$ and the circles $x^2 + y^2 = 2y$, $x^2 + y^2 = 8y$.

Use the transformation

$$u = \frac{2x}{x^2 + y^2}, \quad v = \frac{2y}{x^2 + y^2}$$

to evaluate the integral

$$\iint_R \frac{1}{(x^2 + y^2)^2} dx dy.$$

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