242 Homework #12 Solutions

13.6/3

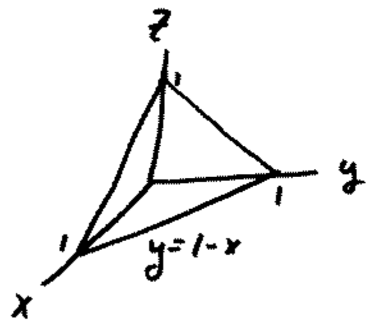
$$\int_{x=-1}^{x=3}\int_{y=0}^{y=2}\int_{z=-2}^{z=6} xyz\,dz\,dy\,dx$$

$$= \int_{x=-1}^{x=3}\int_{y=0}^{y=2} \tfrac{1}{2}xyz^2\Big|_{z=-2}^{z=6}\,dy\,dx$$

$$= \int_{x=-1}^{x=3}\int_{y=0}^{y=2} 16xy\,dy\,dx = \int_{x=1}^{x=3} 8xy^2\Big|_{y=0}^{y=2}\,dx$$

$$= \int_1^3 32x\,dx = 16x^2\Big|_1^3 = 16(9-1) = 128$$

13.6/5



$0 \le z \le 1-x-y$
$0 \le y \le 1-x$
$0 \le x \le 1$

$$\int_{x=0}^{x=1}\int_{y=0}^{y=1-x}\int_{z=0}^{z=1-x-y} x^2\,dz\,dy\,dx$$

$$= \int_{x=0}^{x=1}\int_{y=0}^{y=1-x} x^2(1-x-y)\,dy\,dx$$

→

13.6/5 cont'd

$$= \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} x^2 - x^3 - x^2 y \; dy \, dx$$

$$= \int_{x=0}^{x=1} x^2 y - x^3 y - \tfrac{1}{2} x^2 y^2 \Big|_{y=0}^{y=1-x} dx$$

$$= \int_{x=0}^{x=1} x^2(1-x) - x^3(1-x) - \tfrac{1}{2} x^2 (1 - 2x + x^2) \quad dx$$

$$= \int_0^1 x^2 \cancel{-x^3} - x^3 + x^4 - \tfrac{1}{2} x^2 \cancel{+x^3} - \tfrac{1}{2} x^4 \quad dx$$

$$= \int_0^1 \tfrac{1}{2} x^4 - x^3 + \tfrac{1}{2} x^2 \, dx = \tfrac{1}{10} x^5 - \tfrac{1}{4} x^4 + \tfrac{1}{6} x^3 \Big|_0^1$$

$$= \frac{1}{10} - \frac{1}{4} + \frac{1}{6} = \frac{6 - 15 + 10}{60} = \frac{1}{60}$$

13.6/9 Intersection of surfaces: $\cancel{y^2 = 8 - y^2;}$

$\cancel{2y^2 = 8 \quad y^2 = 4 \quad y = \pm 2}$ $\quad 2 - x^2 = x^2$; $2 = 2x^2$; $x = \pm 1$

$2 - x^2 \le z \le x^2 \qquad -1 \le x \le 1, \qquad 0 \le y \le 3$

should be switched, so I put a -sign in front.

$$- \int_{x=-1}^{x=1} \int_{y=0}^{y=3} \int_{z=2-x^2}^{z=x^2} x + y \quad dz \, dy \, dx \qquad \rightarrow$$

13.6/9 cont'd

$$= -\int_{x=-1}^{x=1}\int_{y=0}^{y=3} xz + yz \Big|_{z=2-x^2}^{z=x^2} \, dy\, dx$$

$$= -\int_{x=-1}^{x=1}\int_{y=0}^{y=3} x^3 + yx^2 - \left[2x - x^3 + 2y - yx^2\right] dy\, dx$$

$$= -\int_{x=-1}^{x=1}\int_{y=0}^{y=3} 2x^3 + 2yx^2 - 2x - 2y \quad dy\, dx$$

$$= -\int_{x=-1}^{x=1} 2x^3 y + y^2x^2 - 2xy - y^2 \Big|_{y=0}^{y=3} dx$$

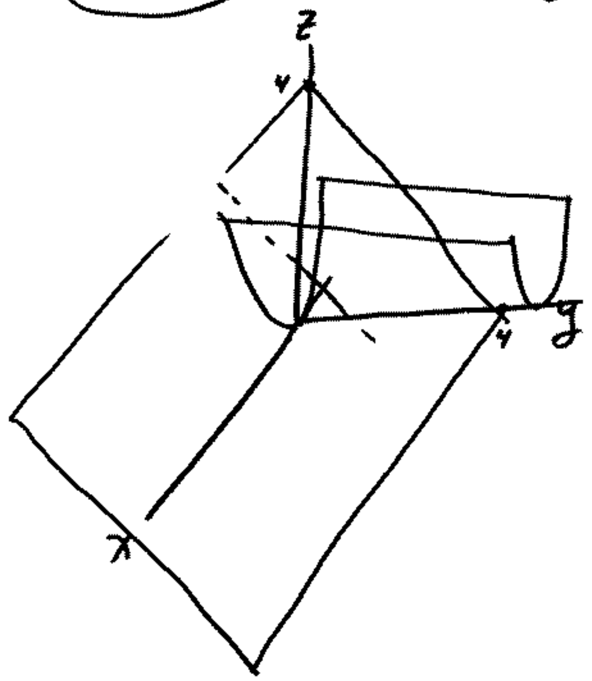
$$= -\int_{x=-1}^{x=1} 6x^3 + 9x^2 - 6x - 9 \; dx$$

$$= -\left[\tfrac{3}{2}x^4 + 3x^3 - 3x^2 - 9x \Big|_{-1}^{1}\right]$$

$$= -\left[\tfrac{3}{2} + 3 - 3 - 9 - \left(\tfrac{3}{2} - 3 - 3 + 9\right)\right]$$

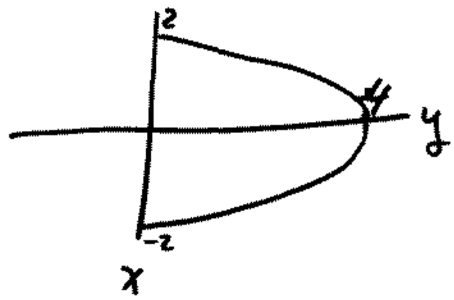
$$= -\left[\cancel{\tfrac{3}{2}} + \cancel{3} - \cancel{3} - 9 - \cancel{\tfrac{3}{2}} + 3 + 3 - 9\right] = -\left[6 - 18\right] = +12$$

(13.6/17) $z = x^2 \quad y + z = 4 \quad y = 0 \quad z = 0$



Find intersection of $z = x^2$ and $y + z = 4$ $(z = 4 - y)$

$x^2 = 4 - y \qquad y = 4 - x^2$



$-2 \le x \le 2$

$0 \le y \le 4 - x^2$

$x^2 \le z \le 4 - y$

$$\int_{x=-2}^{x=2} \int_{y=0}^{y=4-x^2} \int_{z=x^2}^{z=4-y} 1 \, dz \, dy \, dx = \text{Volume}$$

$$= \int_{x=-2}^{x=2} \int_{y=0}^{y=4-x^2} 4 - y - x^2 \, dy \, dx$$

$$= \int_{x=-2}^{x=2} 4y - \tfrac{1}{2}y^2 - x^2 y \Big|_{y=0}^{y=4-x^2} dx$$

$$= \int_{x=-2}^{x=2} 4(4-x^2) - \tfrac{1}{2}(4-x^2)^2 - x^2(4-x^2) \, dx \qquad \rightarrow$$

13.6/17 cont'd

$$= \int_{x=-2}^{x=2} 16-4x^2-\left(8-4x^2+\tfrac{1}{2}x^4\right)-4x^2+x^4\,dx$$

$$= \int_{x=-2}^{x=2} \tfrac{1}{2}x^4-4x^2+8\,dx = \tfrac{1}{10}x^5-\tfrac{4}{3}x^3+8x\Big|_{-2}^{2}$$

$$= \frac{32}{10}-\frac{32}{3}+16-\left(-\frac{32}{10}+\frac{32}{3}-16\right)$$

$$= 2\left(\frac{16}{5}-\frac{32}{3}+16\right) = 2\cdot\frac{48-160+240}{15} = 2\cdot\frac{128}{15} = \frac{256}{15}$$

13.6/23

$$\bar{x} = \int_{x=-2}^{x=2}\int_{y=0}^{y=4-x^2}\int_{z=x^2}^{z=4-y} x\,dz\,dy\,dx$$

(x just carries along until 3rd integration)

$$= \int_{x=-2}^{x=2} \tfrac{1}{2}x^5-4x^3+8x\,dx = \tfrac{1}{12}x^6-x^4+4x^2\Big|_{-2}^{2} = 0$$

$$\bar{y} = \int_{x=-2}^{x=2}\int_{y=0}^{y=4-x^2}\int_{z=x^2}^{z=4-y} y\,dz\,dy\,dx$$

$$= \int_{x=-2}^{x=2}\int_{y=0}^{y=4-x^2} 4y-y^2-x^2y\,dy\,dx \quad \rightarrow$$

13.6/36

$$\text{mass} = m = \int_{x=0}^{x=1}\int_{y=0}^{y=1}\int_{z=0}^{z=1} kz\,dz\,dy\,dx$$

$$= \int_{x=0}^{x=1}\int_{y=0}^{y=1} \frac{k}{2}z^2\Big|_0^1\,dy\,dx$$

$$= \int_{x=0}^{x=1}\int_{y=0}^{y=1} \frac{k}{2}\,dy\,dx = \int_{x=0}^{x=1}\frac{k}{2}\,dx = k/2$$

$$\bar{x} = \frac{2}{k}\int_{x=0}^{x=1}\int_{y=0}^{y=1}\int_{z=0}^{z=1} k\,xz\,dz\,dy\,dx$$

$$= 2\int_{x=0}^{x=1}\int_{y=0}^{y=1} x\cdot\frac{1}{2}z^2\Big|_{z=0}^{z=1}\,dy\,dx = \int_{x=0}^{x=1}\int_{y=0}^{y=1} x\,dy\,dx$$

$$= \int_{x=0}^{x=1} x\,dx = \frac{1}{2}x^2\Big|_0^1 = \boxed{\frac{1}{2}}.$$

$$\bar{y} = \frac{2}{k}\int_{x=0}^{x=1}\int_{y=0}^{y=1}\int_{z=0}^{z=1} k\,yz\,dz\,dy\,dx$$

$$= 2\int_{x=0}^{x=1}\int_{y=0}^{y=1} y\cdot\frac{1}{2}z^2\Big|_{z=0}^{z=1}\,dy\,dx = \int_{x=0}^{x=1}\int_{y=0}^{y=1} y\,dy\,dx$$

$$= \int_{x=0}^{x=1}\frac{1}{2}y^2\Big|_0^1\,dx = \int_0^1 \frac{1}{2}\,dx = \boxed{1/2}$$

→

13.6/36 cont'd

$$\bar{z} = \frac{2}{k}\int_{x=0}^{x=1}\int_{y=0}^{y=1}\int_{z=0}^{z=1} kz^2\,dz\,dy\,dx$$

$$= 2\int_{x=0}^{x=1}\int_{y=0}^{y=1} \frac{1}{3}z^3\Big|_0^1\,dy\,dx = \frac{2}{3}\int_{x=0}^{x=1}\int_{y=0}^{y=1} 1\,dy\,dx = \boxed{\frac{2}{3}}$$

centroid $\boxed{(\bar{x},\bar{y},\bar{z}) = \left(\frac{1}{2},\ \frac{1}{2},\ \frac{2}{3}\right)}$

13.6/37

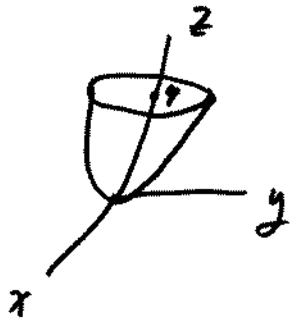
$$I_z = \int_{x=0}^{x=1}\int_{y=0}^{y=1}\int_{z=0}^{z=1}(x^2+y^2)kz\,dz\,dy\,dx$$

$$= k\int_{x=0}^{x=1}\int_{y=0}^{y=1}(x^2+y^2)\frac{1}{2}z^2\Big|_0^1\,dy\,dx$$

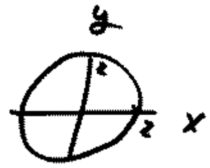
$$= \frac{k}{2}\int_{x=0}^{x=1}\int_{y=0}^{y=1} x^2+y^2\,dy\,dx = \frac{k}{2}\int_{x=0} x^2y+\frac{1}{3}y^3\Big|_{y=0}^{y=1}\,dx$$

$$= \frac{k}{2}\int_{x=0}^{x=1} x^2+\frac{1}{3}\,dx = \frac{k}{2}\left(\frac{1}{3}x^3+\frac{1}{3}x\right)\Big|_0^1 = \frac{k}{2}\cdot\frac{2}{3} = \boxed{\frac{k}{3}}$$

13.7/1



the region is bounded by $r^2 = 4$, $r = 2$



$0 \le \theta \le 2\pi$
$0 \le r \le 2$
$r^2 \le z \le 4$

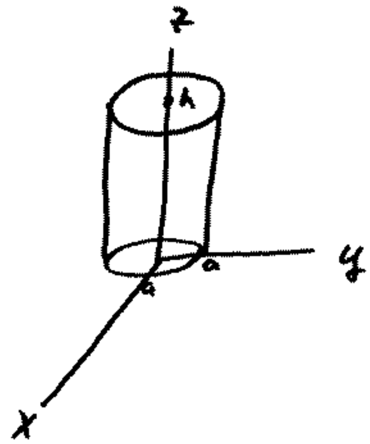
$$Vol = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} \int_{z=r^2}^{z=4} 1 \, r \, dz \, dr \, d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} rz \Big|_{z=r^2}^{z=4} dr \, d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} 4r - r^3 \, dr \, d\theta = \int_{\theta=0}^{\theta=2\pi} 2r^2 - \frac{1}{4} r^4 \Big|_{r=0}^{r=2} d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} 8-4 \, d\theta = \boxed{8\pi}$$

13.7/7



$0 \le \theta \le 2\pi$

$0 \le r \le a$

$0 \le z \le h$

$\delta = z$

$$m = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=a} \int_{z=0}^{z=h} z\, r \; dz\, dr\, d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=a} \frac{1}{2} z^2 r \Big|_{z=0}^{z=h} dr\, d\theta = \frac{h^2}{2} \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=a} r \; dr\, d\theta$$

$$= \frac{h^2}{2} \int_{\theta=0}^{\theta=2\pi} \frac{1}{2} r^2 \Big|_{\phi=0}^{r=a} d\theta = \frac{h^2 a^2}{4} \int_0^{2\pi} 1 \; d\theta$$

$$= \frac{h^2 a^2}{4} \cdot 2\pi = \boxed{\frac{h^2 a^2 \pi}{2}}$$

13.7/15 Find the intersection of the surfaces.

$$x^2+y^2+z^2=2$$

$$z=x^2+y^2$$

$x^2+y^2+(x^2+y^2)^2=2$; $(x^2+y^2)^2+(x^2+y^2)-2=0$

$$(x^2+y^2+2)(x^2+y^2-1)=0$$

$x^2+y^2=-2$ (impossible) or $\boxed{x^2+y^2=1}$

The region in the x-y plane is a circle of radius 1.

$0\le\theta\le 2\pi$, $0\le r\le 1$,

$$r^2=x^2+y^2\le z\le\sqrt{2-x^2-y^2}=\sqrt{2-r^2}$$

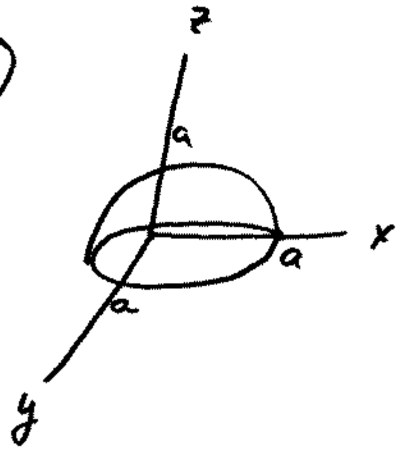
$$Vol=\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1}\int_{z=r^2}^{z=\sqrt{2-r^2}} 1\cdot r\,dz\,dr\,d\theta$$

$$=\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1}(2-r^2)^{\frac{1}{2}}r-r^3\,dr\,d\theta$$

$$=\int_{\theta=0}^{\theta=2\pi}\frac{-1}{2}\cdot\frac{2}{3}(2-r^2)^{3/2}-\frac{1}{4}r^4\Big|_{r=0}^{r=1}d\theta$$

$$=\int_{\theta=0}^{\theta=2\pi}-\frac{1}{3}-\frac{1}{4}-\left(\frac{-1}{3}\cdot 2^{3/2}\right)d\theta=\boxed{2\pi\left(\frac{2^{3/2}}{3}-\frac{7}{12}\right)}$$

13.7/21



In spherical coords.,

$0 \le \rho \le a$

$0 \le \theta \le 2\pi$

$0 \le \phi \le \pi/2$

$z = \rho \cos\phi$

By symmetry, $\bar{x} = \bar{y} = 0$.

$$\bar{z} = \frac{1}{V}\int_{\theta=0}^{\theta=2\pi}\int_{\phi=0}^{\phi=\pi/2}\int_{\rho=0}^{\rho=a} \rho^3 \sin\phi\cos\phi \, d\rho\, d\phi\, d\theta$$

$$= \frac{1}{V}\int_{\theta=0}^{\theta=2\pi}\int_{\phi=0}^{\phi=\pi/2} \frac{1}{4}\rho^4\Big|_{\rho=0}^{\rho=a} \sin\phi\cos\phi\, d\phi\, d\theta$$

$$= \frac{1}{V}\frac{1}{4}a^4 \int_{\theta=0}^{\theta=2\pi}\int_{\phi=0}^{\phi=\pi/2} \sin\phi\cos\phi\, d\phi\, d\theta$$

$$= \frac{1}{V}\frac{a^4}{4}\int_{\theta=0}^{\theta=2\pi} \frac{1}{2}\sin^2\phi\Big|_0^{\pi/2} d\theta = \frac{1}{V}\frac{a^4}{4}\int_{\theta=0}^{\theta=2\pi} \frac{1}{2}\, d\theta$$

$$= \frac{1}{V}\frac{a^4\pi}{4}$$

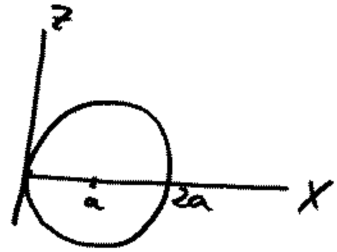
Now Volume $= V = \frac{1}{2}\cdot\frac{4}{3}\pi a^3 = \frac{2}{3}\pi a^3$

$$= \frac{3}{2\pi a^3}\,\frac{a^4\pi}{4} = \frac{3a}{8}.$$

$\boxed{\text{Centroid} = (0, 0, \tfrac{3a}{8})}$

$\boxed{13.7/29}$ It is the circle $(x-a)^2+z^2=a^2$

in the xz-plane, revolved around the z-axis



$0 \le \theta \le 2\pi$

$0 \le \phi \le \pi/2$ ← top half only

$0 \le \rho \le 2a\sin\phi$

$$\text{Vol.} = 2\int_{\theta=0}^{\theta=2\pi}\int_{\phi=0}^{\phi=\pi/2}\int_{\rho=0}^{\rho=2a\sin\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

because $0 \le \phi \le \pi/2$ gives top half only

$$= 2\int_{\theta=0}^{\theta=2\pi}\int_{\phi=0}^{\phi=\pi/2} \frac{1}{3}\rho^3 \Big|_{\rho=0}^{\rho=2a\sin\phi} \sin\phi \, d\phi \, d\theta$$

$$= 2\int_{\theta=0}^{\theta=2\pi}\int_{\phi=0}^{\phi=\pi/2} \frac{1}{3}\cdot 8a^3 \sin^4\phi \, d\phi \, d\theta$$

$$= 2\cdot\frac{8}{3}a^3 \int_{\theta=0}^{\theta=2\pi} \frac{1\cdot 3}{2\cdot 4}\frac{\pi}{2} \, d\theta$$

formula 113 inside back cover

$$= 2\cdot\frac{8}{3}a^3\cdot\frac{3}{16}\pi\cdot 2\pi = 2a^3\pi^2$$