242 Homework #9 Solutions

12.7/20 Implicit differentiation

$$x^3 + y^3 + z^3 = xyz$$

Think of x, y as variables, z as a function of x and y.

$$\frac{\partial}{\partial x}(x^3+y^3+z^3) = \frac{\partial}{\partial x}(xyz)$$

$$3x^2 + 0 + 3z^2\frac{\partial z}{\partial x} = 1\cdot yz + xy\frac{\partial z}{\partial x}$$

Solve for $\partial z/\partial x$:

$$(3z^2 - xy)\frac{\partial z}{\partial x} = yz - 3x^2 \qquad \boxed{\frac{\partial z}{\partial x} = \frac{yz-3x^2}{3z^2-xy}}$$

$$\frac{\partial}{\partial y}(x^3+y^3+z^3) = \frac{\partial}{\partial y}(xyz)$$

$$3y^2 + 3z^2\frac{\partial z}{\partial y} = x\cdot 1\cdot z + xy\frac{\partial z}{\partial y}$$

Solve for $\partial z/\partial y$:

$$(3z^2 - xy)\frac{\partial z}{\partial y} = xz - 3y^2 \qquad \boxed{\frac{\partial z}{\partial y} = \frac{xz-3y^2}{3z^2-xy}}$$

12.8/32

$$3x^2 + 4y^2 + 5z^2 = 73 \qquad P(2,2,3)$$

(This is an ellipsoid.)

$$F(x,y,z) = 3x^2 + 4y^2 + 5z^2 - 73$$

$\nabla F = \langle 6x, 8y, 10z \rangle$ = normal to tangent plane
$= \langle 12, 16, 30 \rangle$ at $P(2,2,3)$

Equation of tangent plane:

$$\cancel{\langle 6x, 8y, 10z \rangle \cdot \langle x-2, y-2, z-3 \rangle = 0}$$

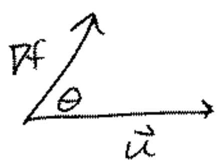
$$\langle 12, 16, 30 \rangle \cdot \langle x-2, y-2, z-3 \rangle = 0$$

$$\boxed{12(x-2) + 16(y-2) + 30(z-3) = 0}$$

12.8/39

The directional derivative $D_{\vec{u}} f$ gives the rate of change of f in the direction of the unit vector $\vec{u}$.

$$D_{\vec{u}} f = \nabla f \cdot \vec{u} = |\nabla f|\,|\vec{u}| \cos\theta = |\nabla f| \cos\theta$$



This is the smallest (i.e. most negative) when $\cos\theta = \pi = 180°$. In this case $\vec{u} = -\nabla f$.

So f decreases most radiply (at P) in the direction $-\nabla f(P)$.

12. 9/6

$f(x,y) = 4x^2 + 9y^2$ $\quad g(x,y) = x^2 + y^2 - 1 = 0$

$\nabla f = \langle 8x, 18y \rangle$ $\quad \nabla g = \langle 2x, 2y \rangle$

Find points (x,y) on $g(x,y)=0$ where $\nabla f = \lambda \nabla g$:

① $x^2 + y^2 - 1 = 0$

② $8x = 2\lambda x$

③ $18y = 2\lambda y$

From ②, $x = 0$ or $\lambda = 4$

If $x=0$, then $y = \pm 1$ from ①

so $(0,1)$, $(0,-1)$ are candidates.

If $\lambda = 4$, then in ③, $18y = 8y$,

so $y=0$. from ①, $x = \pm 1$.

So $(1,0)$, $(-1,0)$ are candidates.

$\nabla g = \vec{0}$ only at $(0,0)$, which is not a point on $g(x,y)=0$.

Check

x	0	0	1	-1
y	1	-1	0	0
$f(x,y)$	9	9	4	4

$\boxed{\text{The max is 9, at } (x,y) = (0,1) \text{ and } (0,-1). \quad \text{The min is 4, at } (x,y) = (1,0) \text{ and } (-1,0).}$

12.9/12

$f(x,y,z) = x - y + z \qquad g(x,y,z) = z - x^2 + 6xy - y^2$

$\nabla f = \langle 1, -1, 1 \rangle$

$\nabla g = \langle -2x + 6y,\ 6x - 2y,\ 1 \rangle$

Find points on $g(x,y,z)$ where $\nabla f = \lambda \nabla g$:

① $z = x^2 + 6xy + y^2$

② $1 = \lambda(-2x + 6y)$

③ $-1 = \lambda(6x - 2y)$

④ $1 = \lambda \cdot 1$

From ④, $\lambda = 1$. So:

②' $1 = -2x + 6y$

③' $-1 = 6x - 2y$

Multiply ②' by 3 and add to ③' to get:

$2 = 16y. \quad \boxed{y = 1/8}$

From ②', $1 = -2x + 3/4$

$1/4 = -2x$

$x = -1/8$

From ①: $z = \frac{1}{64} + \frac{6}{64} + \frac{1}{64} = 1/8$

The candidate for max, min is $(x,y,z) = (-1/8, 1/8, 1/8)$.

Find points on $g(x,y,z) = 0$ where $\nabla g = \vec{0}$.

Note ∇g is never $= \vec{0}$.

cont'd →

12.9/12 cont'd

The only candidate for max or min is $(x,y,z) = (-1/8,\ 1/8,\ 1/8)$

$$f(-1/8, 1/8, 1/8) = \frac{-1}{8} - \frac{1}{8} + \frac{1}{8} = \frac{-1}{8}$$

There are other points on $g(x,y,z)=0$, for example $(0,0,0)$, where f is larger.

So f has a min. of $-1/8$, occurring at $(-\frac{1}{8}, 1/8, 1/8)$ on $g(x,y,z)=0$, and no max.

(You can make f as large as you like by taking $y=0$ and $z=x^2$ very large)

12.9/19 The square of the distance from (x,y) to the origin is $f(x,y) = x^2 + y^2$.

The constraint is $g(x,y) = 3x + 4y - 100 = 0$.

$\nabla f = \langle 2x, 2y \rangle$

$\nabla g = \langle 3, 4 \rangle$

cont'd →

12.9/19 cont'd

We seek a point on $g(x,y)=0$ where $\nabla f = \lambda \nabla g$:

① $3x+4y=100$

② $(2x=3\lambda) \times 4$

③ $(2y=4\lambda) \times 3$

from ② and ③,

$8x=6y$ so $4x=3y$

Put this into ①:

$$3\left(\frac{3}{4}y\right)+4y=100$$

$$\left(\frac{9}{4}+4\right)y=100$$

$$\frac{25}{4}y=100$$

$$y=16$$

then $x=\frac{3}{4}(16)=12$

The only candidate is $(x,y)=(12,16)$.
This must be the point on $3x+4y=100$ closest to the origin, since geometrically it is clear there must <u>be</u> such a point.

12.10/6 $f(x,y) = x^2 + 4xy + 2y^2 + 4x - 8y + 3$

$f_x = 2x + 4y + 4 = 0$

$f_y = 4x + 4y - 8 = 0$

subtract the two equations to get

$-2x + 12 = 0 \quad x = -6$

then from ①,

$-12 + 4y + 4 = 0 \ ; \ 4y = 8 \ ; \ y = 2$

Critical point: $(x,y) = (-6, 2)$.

$f_{xx} = 2 \quad f_{xy} = 4$

$f_{yx} = 4 \quad f_{yy} = 4$

$\boxed{\Delta = 8 - 16 = -8}$

Since $\Delta < 0$, f has neither a local max nor a local min at $(-6,2)$. It is a saddle point.

12.10/10

$$f(x,y) = 3xy - x^3 - y^3$$

$$\left.\begin{array}{lll} f_x = 3y - 3x^2 = 0 & \Rightarrow & y = x^2 \\ f_y = 3x - 3y^2 = 0 & \Rightarrow & x = y^2 \end{array}\right\} \begin{array}{l} x = x^4 \\ \text{so } x = 0 \text{ or} \\ x = 1 \end{array}$$

If $x=0$, then $y=0$. If $x=1$, then $y=1$.

Critical points: $(0,0)$, $(1,1)$.

$$\left.\begin{array}{ll} f_{xx} = -6x & f_{xy} = 3 \\ f_{yx} = 3 & f_{yy} = -6y \end{array}\right\} \Delta = 36xy - 9$$

For $(0,0)$, $\Delta = -9 < 0$, so saddle point.

For $(1,1)$, $\Delta = 36 - 9 > 0$ and $f_{xx} = -6 < 0$, so local max.

$\boxed{\text{There is a local max at } (1,1) \text{ and a saddle point at } (0,0).}$

12.10/20 $\quad f(x,y) = 2x^3 + y^3 - 3x^2 - 12x - 3y$

$f_x = 6x^2 - 6x - 12 = 0 \Rightarrow x^2 - x - 2 = 0;\ (x-2)(x+1) = 0$

$x = 2, -1$

$f_y = 3y^2 - 3 = 0 \Rightarrow y = \pm 1$

Critical points: $(2,1), (2,-1), (-1,1), (-1,-1)$.

$f_{xx} = 12x - 6 \qquad f_{xy} = 0$

$\Delta = (12x-6)(6y)$

$f_{yx} = 0 \qquad f_{yy} = 6y$.

Check:

x	y	Δ	f_{xx}	type
2	1	>0	>0	local min.
2	-1	<0		saddle point
-1	1	<0		saddle point
-1	-1	>0	<0	local max.

12.10/32

$$f(x,y) = x^3 - 3xy^2$$

(a) $\left.\begin{array}{l} f_x = 3x^2 - 3y^2 = 0 \\ f_y = -6xy = 0 \end{array}\right\}$ From $f_y = 0$, either $x=0$ or $y=0$. From $f_x = 0$, $x^2 = y^2$, so $(0,0)$ is the only critical point.

$$\Delta = f_{xx} f_{yy} - f_{xy} f_{yx} = (6x)(-6x) - (-6y)(-6y)$$

$\Delta = 0$ at $(x,y) = (0,0)$

(b) • Consider the line $x=0$. $f(0,y) = 0$. doesn't help.

• Consider the line $y=0$ $f(x,0) = x^3$.

as x increases, $f(x,0)$ increases

as x decreases, $f(x,0)$ decreases.

• Consider the line $x=y$. $f(x,y) = x^3 - 3x^3 = -2x^3$

as you go in one direction along this line, f increases and in the other direction, f decreases.

• Consider the line $x=-y$. $f(x,y) = -2x^3$

as you go in one direction along this line f increases; in the other direction, f decreases.

Therefore it is a monkey saddle at $(0,0)$.