Class Notes on Section 11.6

Recall for a parametric curve

$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$, we define

$\vec{v}(t) = \vec{r}'(t)$ velocity

$\vec{a}(t) = \vec{v}'(t) = \vec{r}''(t)$ acceleration

$v(t) = |\vec{v}(t)|$ speed

Arc length along the curve from $t=a$ to $t=t$ is defined $s(t) = \int_a^t v(\tau)\, d\tau$

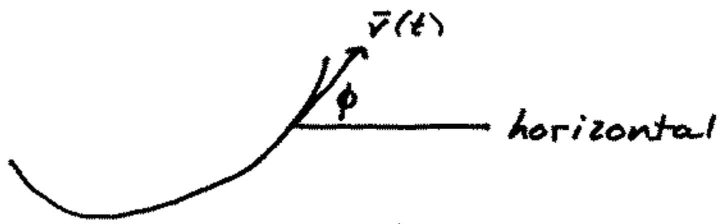
By the Fundamental Theorem of Calculus,

$$\frac{ds}{dt} = v$$

Speed = rate of change of arc length

Curvature of plane curves

$\vec{r}(t) = \langle x(t), y(t) \rangle$.



With ϕ the angle as shown, define curvature

$$K = \left| \frac{d\phi}{ds} \right|$$

Note: K is large if the direction of the curve is changing rapidly

Do some trigonometry and calculus and algebra to derive the following formula, which is used to actually compute κ:

$$\kappa = \frac{|x'y'' - x''y'|}{v^3}.$$

Here $x' = \frac{dx}{dt}$, $x'' = \frac{d^2x}{dt^2}$, $y' = \frac{dy}{dt}$, $y'' = \frac{d^2y}{dt^2}$.

Unit tangent vector + unit normal vector

Let $\vec{T} = \frac{\vec{v}}{|\vec{v}|}$ unit tangent vector.
Same direction as $\vec{v}$, length 1.

$1 = |\bar{T}|^2 = \bar{T} \cdot \bar{T}$, so

$$0 = \frac{d\bar{T}}{ds} \cdot \bar{T} + \bar{T} \cdot \frac{d\bar{T}}{ds} = 2\,\bar{T} \cdot \frac{d\bar{T}}{ds}.$$

Therefore $\frac{d\bar{T}}{ds} \perp \bar{T}$

Let $\bar{N} = \frac{1}{\left|\frac{d\bar{T}}{ds}\right|} \frac{d\bar{T}}{ds}$ = principal unit normal vector.

One can show $\left|\frac{d\bar{T}}{ds}\right| = \kappa$, so then

$$\frac{d\bar{T}}{ds} = \kappa \vec{N}$$

Curvature of space curves

As for plane curves, we define the unit tangent vector $\vec{T} = \frac{\vec{v}}{|\vec{v}|} = \frac{\vec{v}}{v}$

so $\vec{v} = v\vec{T}$

How to define κ? We don't have a convenient angle ϕ as for plane curves.

Instead, note $\frac{d\vec{T}}{ds} \perp \vec{T}$ (same computation as for plane curves)

Define $\vec{N} = \frac{1}{\left|\frac{d\vec{T}}{ds}\right|}\frac{d\vec{T}}{ds}$

Define curvature $\kappa = \left|\frac{d\vec{T}}{ds}\right|$

Note: $\left|\frac{d\vec{T}}{ds}\right| = \left|\frac{d\vec{T}}{dt}\right|\left|\frac{dt}{ds}\right| = \left|\frac{d\vec{T}}{dt}\right| \div \left|\frac{ds}{dt}\right| = \frac{1}{v}\left|\frac{d\vec{T}}{dt}\right|$

so $\kappa = \frac{1}{v}\left|\frac{d\vec{T}}{dt}\right|$ (more convenient for computing)

Note $\frac{d\vec{T}}{ds} = \kappa\vec{N}$

Interpreting the acceleration vector

Note: $v = |\vec{v}| =$ speed, but

$|\vec{a}|$ is not in general $\frac{dv}{dt}$

An example is $\vec{r}(t) = \langle \cos t, \sin t \rangle$.

$\vec{a}$ includes the change in speed $\frac{dv}{dt}$ (tangential component) AND the change in direction (normal component).

Calculate:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(v\vec{T}) = \frac{dv}{dt}\vec{T} + v\frac{d\vec{T}}{dt}$$

$$= \frac{dv}{dt}\vec{T} + v\frac{d\vec{T}}{ds} \cdot \frac{ds}{dt}$$

Now $\frac{ds}{dt} = v$ and $\frac{d\vec{T}}{ds} = \kappa \vec{N}$ from before.

so $$\boxed{\vec{a} = \frac{dv}{dt}\vec{T} + \kappa v^2 \vec{N}}$$

Define $a_T = \frac{dv}{dt}$ tangential component of acceleration

$a_N = \kappa v^2$ normal component of acceleration

We can write

$$\vec{a} = a_T \vec{T} + a_N \vec{N}$$

a_T is the rate of change of speed

a_N is the rate of change of direction with respect to time.

If the particle moves at a constant speed along any curve, then $a_T = 0$

If the particle moves at any speed along a straight line, then $a_N = 0$.

The following formulas can be derived. They are used for actually computing a_T and a_N.

$$a_T = \frac{\vec{v} \cdot \vec{a}}{v} = \frac{\vec{r}\,' \cdot \vec{r}\,''}{|\vec{r}\,'|}$$

$$a_N = \frac{|\vec{v} \times \vec{a}|}{v} = \frac{|\vec{r}\,' \times \quad|}{|\vec{r}\,'|}$$

This class will not cover pp. 827-830 on Newton + Kepler's laws of planetary motion.