Name:____

Math 247, Section F1 - Test #1 - February 18, 2000

Time: 50 minutes. You may not use any books or notes. A calculator is not necessary, but is permitted. Partial credit will be given based on what it written, not what is intended. There are 100 points total.

1. (15 points) Write the negation of each statement:

   (a) For all $x \in \mathbf{R}$, there exists $y \in \mathbf{R}$ such that $xy = 1$.

   There exists an $x \in \mathbb{R}$ such that for all $y \in \mathbb{R}$, $xy \neq 1$.

   (b) For all pairs $a_1 \in A$ and $a_2 \in A$, if $a_1 > a_2$, then $f(a_1) > f(a_2)$.

   There exists a pair $a_1 \in A$ and $a_2 \in A$ such that $a_1 > a_2$ and $f(a_1) \leq f(a_2)$.

   (c) The function f is increasing on the interval $(0, 1)$ or it is decreasing on the interval $(0, 1)$.

   The function f is not increasing on the interval $(0,1)$ and it is not decreasing on the interval $(0,1)$.

   Note: "not increasing" does not mean the same as "non-increasing".

2. (15 points) Prove by contrapositive:

If $\frac{x}{y} + \frac{y}{x} \geq 2$, then $xy > 0$.

Suppose $xy \leq 0$.

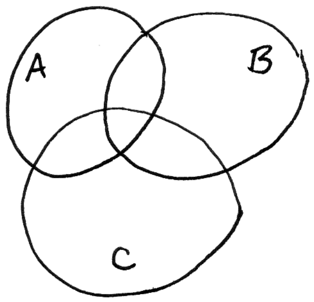
If $xy = 0$, then $\frac{x}{y} + \frac{y}{x} \geq 2$ is certainly false, since either x or y is zero, so either $\frac{x}{y}$ or $\frac{y}{x}$ is undefined.

If $xy < 0$, then either $x > 0$ and $y < 0$ or $x < 0$ and $y > 0$. In both cases, $\frac{x}{y}$ and $\frac{y}{x}$ are both negative, so $\frac{x}{y} + \frac{y}{x}$ is negative and $\frac{x}{y} + \frac{y}{x} \geq 2$ is false.

We have shown that if $xy \leq 0$, then $\frac{x}{y} + \frac{y}{x} \geq 2$ is false. Therefore, if $\frac{x}{y} + \frac{y}{x} \geq 2$, then $xy > 0$.

3. (15 points) Decide whether the following statement is true or false and either prove it or give a counterexample to disprove it (A Venn diagram may be helpful, but is not a sufficient answer by itself):

For all sets A, B and C, $(A \cap B) \cup C = A \cap (B \cup C)$.



This is false.
For example, let
$A = \{1, 2\}$, $B = \{2, 3\}$
and $C = \{2, 4\}$.

Then $A \cap B = \{2\}$ and $(A \cap B) \cup C = \{2, 4\}$.

$B \cup C = \{2, 3, 4\}$, and $A \cap (B \cup C) = \{2\}$.

We see $(A \cap B) \cup C \neq A \cap (B \cup C)$ for this example.

4. (a) (10 points) Let $f : A \to B$. Define (i) f is *injective*, (ii) f is *surjective*, (iii) f is *bijective*.

(i) For every $b \in B$, there is at most one $a \in A$ such that $f(a) = b$.

(ii) For every $b \in B$, there is at least one $a \in A$ such that $f(a) = b$.

(iii) For every $b \in B$, there is exactly one $a \in A$ such that $f(a) = b$.

(b) (10 points) Give a bijection from the set of even integers to **N**. The bijection should be clearly described, but a formula is optional. Don't forget the negative even integers.

even integers		$\mathbb{N}$
0	$\longrightarrow$	1
2	$\longrightarrow$	2
−2	$\longrightarrow$	3
4	$\longrightarrow$	4
−4	$\longrightarrow$	5
6	$\longrightarrow$	6
−6	$\longrightarrow$	7

etc.

A formula (optional) for this bijection is for each even integer x,

$$f(x) = \begin{cases} x & \text{if } x > 0 \\ -x+1 & \text{if } x \leq 0. \end{cases}$$

5. (20 points) Prove that for all $n \in \mathbf{N}$,

$$\sum_{i=1}^{n} \frac{i}{2^i} = 2 - \frac{n+2}{2^n}.$$

The proof is by induction on n. Let $P(n)$ be the statement $\sum_{i=1}^{n} \frac{i}{2^i} = 2 - \frac{n+2}{2^n}$.

$P(1)$ is $\frac{1}{2^1} = 2 - \frac{1+2}{2}$. Since $2 - \frac{3}{2} = \frac{1}{2}$, $P(1)$ is true.

Suppose $P(k)$ is true: $\sum_{i=1}^{k} \frac{i}{2^i} = 2 - \frac{k+2}{2^k}$.

Add $\frac{k+1}{2^{k+1}}$ to both sides to get

$$(*) \quad \sum_{i=1}^{k+1} \frac{i}{2^i} = 2 - \frac{k+2}{2^k} + \frac{k+1}{2^{k+1}}.$$

Since

$$-\frac{k+2}{2^k} + \frac{k+1}{2^{k+1}} = \frac{-2(k+2) + k+1}{2^{k+1}} = \frac{-k-3}{2^{k+1}} = -\frac{(k+1)+2}{2^{k+1}},$$

we can rewrite $(*)$ as

$$\sum_{i=1}^{k+1} \frac{i}{2^i} = 2 - \frac{(k+1)+2}{2^{k+1}},$$ which is $P(k+1)$.

By mathematical induction, $P(n)$ is true $\forall n \in \mathbb{N}$.

6. (15 points) Prove that if $x + \epsilon > y$ for all $\epsilon > 0$, then $x \geq y$.

The proof is by contradiction.

Suppose $x < y$. Then $y - x > 0$.

Let $\varepsilon = y - x$. $\varepsilon > 0$.

Then $x + \varepsilon = x + (y - x) = y$, which contradicts $x + \varepsilon > y \quad \forall \varepsilon > 0$.

So if $x + \varepsilon > y \quad \forall \varepsilon > 0$, then $x \geq y$.