Name:____

Math 247, Section F1 - Test #2 - March 24, 2000

Note: the points add to more than 100 because I have combined problems from several exams that I gave in past semesters for this practice exam Fall 2002.

1. (15 points) Prove that if $g : A \to B$ is an injection and $f : B \to C$ is an injection, then $f \circ g$ is an injection.

   This is most easily proved using the statement that a function h is injective if and only if $h(x) = h(y)$ implies $x = y$. Suppose $f \circ g(a_1) = f \circ g(a_2)$. Then $f(g(a_1)) = f(g(a_2))$. Since f is injective, $g(a_1) = g(a_2)$. Since g is injective, $a_1 = a_2$. Therefore $f \circ g$ is injective.

2. (15 points) Out of 100 senators, an ethics committee of 5 will be chosen, with one of the committee members designated as chairperson. How many ways are there to do this?

   Out of the 100 senators, choose 5 to be on the committee. Out of these 5, choose one to be chairperson. The number of ways to do this is

$$\binom{100}{5}\binom{5}{1} = \frac{100!}{95!5!} \cdot \frac{5!}{4!1!} = \frac{100 \cdot 99 \cdot 98 \cdot 97 \cdot 96}{4 \cdot 3 \cdot 2} = 376,437,600.$$

3. Let $d = \gcd(5891, 6987)$.

   (a) (10 points) Find d.

   (b) (10 points) Find integers m and n such that $5891m + 6987n = d$.

   $d = 137$. To get this, use the Euclidean Algorithm:

$$\begin{array}{rl}
(5891, 6987) & 1096 = 6987 - 5891 \\
(5891, 1096) & 411 = 5891 - 5(1096) \\
(1096, 411) & 274 = 1096 - 2(411) \\
(411, 274) & 137 = 411 - 274 \\
(274, 137) & 0 = 274 - 2(137)
\end{array}$$

   To answer part (b), work backwards:

$$\begin{aligned}
137 &= 411 - 274 \\
&= 411 - (1096 - 2(411)) = 3(411) - 1096 \\
&= 3(5891 - 5(1096)) - 1096 = 3(5891) - 16(1096) \\
&= 3(5891) - 16(6987 - 5891) = 19(5891) - 16(6987).
\end{aligned}$$

   Therefore $m = 19$ and $n = -16$.

4. (15 points) Prove that for all $a, b \in \mathbf{Z}$, $\gcd(a, b) = \gcd(a - 2b, b)$. (There is a more general theorem which implies this, but I am asking for a proof in this specific case.)

   We will show that the set of common divisors of a and b is the same as the set of common divisors of $a - 2b$ and b. From this we can conclude $\gcd(a, b) = \gcd(a - 2b, b)$.

   Let x be a common divisor of a and b. Then x is a divisor of $-2b$, so x is a divisor of the sum $a + (-2b)$. Therefore x is a common divisor of $a - 2b$ and b. Now suppose that y is a common divisor of $a - 2b$ and b. Then y is a divisor of $2b$, and y is a divisor of the sum $(a - 2b) + 2b$, so y divides a. Therefore y is a common divisor of a and b. The proof is complete.

5. (15 points) Let $A \subseteq \mathbf{R}$, $B \subseteq \mathbf{R}$ and let $f : A \to B$ be a bijection. Prove that if f is increasing on A, then f^{-1} is increasing on B.

   Let $y_2, y_1 \in B$, $y_2 > y_1$. Let $x_2 = f^{-1}(y_2)$ and $x_1 = f^{-1}(y_1)$. By definition of f^{-1}, $f(x_2) = y_2$ and $f(x_1) = y_1$. We suppose that $x_2 \leq x_1$ and obtain a contradiction. If $x_2 \leq x_1$, then, since f is increasing, $f(x_2) \leq f(x_1)$, so $y_2 \leq y_1$, which contradicts $y_2 > y_1$. Therefore our assumption that $x_2 \leq x_1$ is false; $x_2 > x_1$, so $f^{-1}(y_2) > f^{-1}(y_1)$ whenever $y_2 > y_1$, so f^{-1} is increasing.

6. (20 points) Prove that $\sqrt{13}$ is irrational.
   Note: There is a theorem which says "The positive integer k has no rational square root if k is not the square of an integer." If you choose to use this theorem in your proof, you must give a proof of the theorem first.

   Suppose that $\sqrt{13}$ is rational. Then there exist relatively prime positive integers p and q such that $\sqrt{13} = p/q$. (We can always arrange that p and q be relatively prime by cancelling any common factors) This implies $13q^2 = p^2$. Therefore 13 divides p^2. Since 13 is prime, this means 13 divides p. We write $p = 13k$ for some integer k. Substituting into $13q^2 = p^2$ and cancelling gives us $q^2 = 13k^2$. Now we see that 13 divides q^2, so 13 divides q. Thus 13 is a common divisor of p and q, which contradicts p and q relatively prime. Therefore $\sqrt{13}$ is not rational.

7. (10 points) Suppose that x is irrational, a is rational and $a \neq 0$. Prove that $ax + a$ must be irrational.

   The proof is by contradiction. Suppose that $ax + a$ is rational. Then, since a is rational and the difference of two rational numbers is rational. $(ax+a)-a = ax$ is rational. The quotient of two rational numbers is rational (as long as the denominator is not 0), so $(ax)/a = x$ is rational. This contradicts the hypothesis that x is irrational, so we conclude that $ax + a$ is irrational.

8. (10 points) Give an example to show that the statement "If x is irrational and

y is irrational, then xy must be irrational." is *false*.

One example is $x = \sqrt{2}$ and $y = \sqrt{2}$. Both are irrational, but their product, 2, is rational.