

Planar Wallpaper Symmetry Groups

A finite figure with mirror symmetry has dihedral \mathcal{D}_n symmetry for some n , meaning that there are n mirror lines meeting at equal angles of π/n .

In a wallpaper pattern, meaning a figure that repeats infinitely in different directions, there is always translational symmetry. To classify its further symmetries (if any), you should first look for mirror symmetry. Any mirror line will be part of an infinite family of parallel mirror lines. Look carefully in between two you have drawn to make sure you find them all.

At a point where n mirror lines meet, they do so at equal π/n angles as in \mathcal{D}_n . A kaleidoscope is a mirrored box: some polygon in the plane with mirrors along each edge. To see nice symmetric patterns in the kaleidoscope, we want the two mirrors meeting at each corner of the polygon to do so at one of the allowed angles π/n . There are only four possibilities for a closed polygon with these angles: right triangles with angles $\pi/6$, $\pi/3$, $\pi/2$ or $\pi/4$, $\pi/4$, $\pi/2$, an equilateral triangle with three $\pi/3$ angles, or a square or rectangle with equal $\pi/2$ angles. These are the four possible kinds of kaleidoscopes in two dimensions. When a pattern has exactly the mirror symmetry of a kaleidoscope, we label the symmetry with a \star for the mirror followed by numbers listing the corners of the mirror box. Thus we have the four symmetry patterns $\star 632$, $\star 442$, $\star 333$ and $\star 2222$.

The symmetry groups in each of these cases include orientation-preserving and -reversing isometries. We can consider, in each case, the subgroup of only orientation-preserving isometries. These groups are called 632, 442, 333 and 2222. Patterns with these symmetries are slightly harder to recognize. But note that the names describe the different kinds of rotational centers, and these occur in the same geometric pattern as when there are mirror lines.

Now, patterns with the last two kinds of mirror boxes ($\star 333$ and $\star 2222$) can have additional rotational symmetry around the box centers. This symmetry creates an additional rotation center not on any mirror line, and identifies some of the corners of the kaleidoscope. The names for these patterns are $3\star 3$, $2\star 22$ and $4\star 2$.

The remaining symmetry patterns have no mirror corners, either because they have mirror lines only in one direction, or because they have no mirror symmetry at all. If there are mirror lines in one direction (say vertical), there are three possibilities. Either (1) there are two (alternating) essentially different kinds of mirror lines, or all mirror lines are equivalent and either (2) there is a glide line between successive mirrors or (3) there are rotational centers between the mirrors. These three patterns are called $\star\star$, $\times\star$ and $22\star$, respectively, where \times indicates a glide. In the first two cases, there is a well-defined “up” direction along the mirror lines, while in the last case, the rotation exchanges “up” and “down”.

We already listed four symmetry groups with no mirrors, the orientation-preserving parts of the kaleidoscope groups. The other patterns with no mirror symmetry are again of three types. If there is nothing but translational symmetry, we call the pattern \circ . If there are two kinds of glide lines, with no rotational symmetry, we have $\times\times$. Finally, if there are rotational centers between the glide lines, we have $22\times$.

These 17 possibilities we have listed are the only wallpaper groups in the plane. One proof of this fact is based on the following (seemingly amazing) calculation.

Suppose we assign costs to the various symbols in the names: \circ costs \$2, while \times and \star cost \$1 each. Finally a number n costs $\$ \frac{n-1}{n}$, except that after a \star (when it indicates a mirror corner rather than just a rotation center) it only costs half as much. Then the 17 wallpaper groups we have listed are exactly the 17 combinations of these symbols that cost a total of exactly \$2.