Math 315, Section C2 - Test #1 Solutions

1. (13 points) Solve the system of equations. (Use any method, but show your steps. Indicate clearly what your solution is.)

$$\begin{array}{rrrl} x_1 & +x_2 & +x_3 = & 0 \\ x_1 & -x_2 & -x_3 = & 4. \end{array}$$

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 0 \\ 1 & -1 & -1 & 4 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 1 & 1 & 0 \\ 0 & -2 & -2 & 4 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & -2 \end{array}\right].$$

The variable x_3 is free; let $x_3 = \alpha$. From the second equation, $x_2 = -2 - \alpha$. From the first equation, $x_1 = -x_2 - x_3 = 2 + \alpha - \alpha = 2$. The solution is

$$(x_1, x_2, x_3) = (2, -2 - \alpha, \alpha),$$

where α can be any real number.

2. (13 points) Find the determinant of the following matrix. Hint: there is a short way to do this.

$$\begin{bmatrix} 0 & 3 & -2 & 1 \\ 0 & 0 & 0 & 2 \\ 1 & 2 & -1 & 4 \\ 0 & 0 & 5 & 7 \end{bmatrix}$$

If we switch the first and the third rows, then switch the second and fourth rows, then switch the second and third rows, we obtain an upper triangular matrix. Its determinant is the product of the diagonal entries, $1 \cdot 3 \cdot 5 \cdot 2 = 30$. Since we made three row switches and each row switch changes the sign of the determinant but otherwise leaves it unchanged, the determinant of the original matrix is $(-1)^3 30 = -30$.

3. (13 points) Determine whether or not the following matrix is nonsingular. If so, find its inverse. If not, explain.

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left[\begin{array}{rrr|rrr} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{rrr|rrr} 1 & 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right].$$

The matrix is nonsingular and its inverse is

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

4. (10 points each part) Give an example of each of the following. You do not need to give a proof that the example works.

(a) A proper subspace of $\mathbf{R}^3$. (*Proper* means a subspace other than all of $\mathbf{R}^3$.) One example is $\{(x_1, x_2, x_3) : x_1 = 0\}$. There are many other examples.

(b) An inconsistent linear system. One example is $x_1 + x_2 = 1,\ x_1 + x_2 = 2$.

5. (5 points each part) Give the definition of

(a) Symmetric matrix. A symmetric matrix is a matrix A such that $A^T = A$.

(b) (Multiplicative) inverse of an $n \times n$ matrix A. The inverse of A (if it exists) is a matrix B such that $AB = I$ and $BA = I$.

6. (3 points each part) For each of the following, answer True or False. No explanation is needed.

(a) $\{f \in C[-1, 1] : f(0) = 1\}$ is a subspace of the vector space $C[-1, 1]$. **False**. (Neither C1 nor C2 holds).

(b) If $\mathbf{b}$ is a linear combination of the columns of A, then $A\mathbf{x} = \mathbf{b}$ has at least one solution. **True**.

(c) For all $n \times n$ matrices A and B, $\det(AB) = \det(BA)$. **True**. $AB = BA$ is not necessarily true; however, $\det(AB) = \det(A)\det(B)$, from which we may conclude that the given identity is true.

(d) If A is an $n \times n$ matrix with $\det(A) = 0$, then $N(A) = \{\mathbf{0}\}$. **False**. (If $\det(A) = 0$, then by a theorem in Section 1.4, $A\mathbf{x} = \mathbf{0}$ has more than one solution, and by definition, all of these solutions are in $N(A)$).

7. (9 points) Show that the following identity is true for all nonsingular matrices A:

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

By definition of inverse, $AA^{-1} = I$. We know that $\det(I) = 1$ and by a theorem in Section 2.2, $\det(AA^{-1}) = [\det(A)][\det(A^{-1})]$. Therefore $[\det(A)][\det(A^{-1})] = 1$. Since A is nonsingular, its determinant is nonzero, so divide by $\det(A)$ to get

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

8. (10 points) Let V be a vector space. Using only the definition of vector space (provided), show that for all scalars β,

$$\beta\mathbf{0} = \mathbf{0}.$$

By A3 and A5, $\beta\mathbf{0} = \beta(\mathbf{0} + \mathbf{0}) = \beta\mathbf{0} + \beta\mathbf{0}$. By A4, there is an element $-(\beta\mathbf{0})$ such that $\beta\mathbf{0} + (-(\beta\mathbf{0})) = \mathbf{0}$. Add this element to both sides of the previous equation to get $\beta\mathbf{0} + (-\beta\mathbf{0}) = [\beta\mathbf{0} + \beta\mathbf{0}] + [-(\beta\mathbf{0})]$. The left hand side is $\mathbf{0}$, by A4. Using A2, A3 and A4, the right hand side is $\beta\mathbf{0} + [\beta\mathbf{0} + (-(\beta\mathbf{0}))] = \beta\mathbf{0} + \mathbf{0} = \beta\mathbf{0}$. Therefore

$$\mathbf{0} = \beta\mathbf{0}.$$