Math 344, Section X1, Fall 2001

Review Problems for Chapters 3 and 4

Note: This is just a sampling of problems. It does not include everything. The test will be on Monday, Oct. 22 and will cover Sections 3.1-3.5, 3.7, 4.1-4.2.

1. Section 3.1
   (a) Give the definition of "the sequence (x_n) converges to x."
   (b) Prove that if (x_n) converges to x' and if the same sequence (x_n) also converges to x'', then $x' = x''$.
   (c) Give an example of a sequence which converges to 2. Give an example of a sequence that diverges.
   (d) Prove that $((-1)^n)$ diverges (does not converge to any x).
2. Section 3.2
   (a) Give the definition of "bounded sequence".
   (b) Prove that if (x_n) converges, then (x_n) is bounded.
   (c) Give an example of a bounded sequence which diverges.
   (d) Prove that if (x_n) converges to x and if (y_n) converges to y, then $(x_n + y_n)$ converges to $x + y$.
3. Section 3.3
   (a) Give the definitions of "increasing sequence," "decreasing sequence," "monotone sequence."
   (b) Give an example of a monotone sequence which diverges. Give an example of a sequence which converges but which is not monotone.
   (c) State the Monotone Convergence Theorem.
4. Section 3.4
   (a) Give the definition of "subsequence."
   (b) True or false?: If (x_{n_k}) is a subsequence of (x_n) and if (x_{n_k}) converges to x, then (x_n) must converge to x.
   (c) True or false?: If (x_{n_k}) is a subsequence of (x_n) and if (x_n) converges to x, then (x_{n_k}) must converge to x.
   (d) True or false?: If (x_n) diverges, then (x_n) must be unbounded.
   (e) True of false?: If (x_n) is unbounded, then (x_n) diverges.
   (f) True or false?: Every sequence has a convergent subsequence.
   (g) State the Bolzano-Weierstrass Theorem.

5. General question for Chapter 3: List as many ways as you can think of to show that a sequence of real numbers converges. List as many ways as you can think of to show that a sequence of real numbers diverges.

6. Section 3.5

   (a) Give the definition of Cauchy sequence.

   (b) Prove that $(\frac{1}{n})$ is a Cauchy sequence.

   (c) Prove that a convergent sequence must be Cauchy.

   (d) True or false?: A Cauchy sequence (of real numbers) must converge.

7. Section 3.7

   (a) What is the difference between a sequence and a series?

   (b) Give the definition of partial sums.

   (c) Give an example of a series that converges.

   (d) Give an example of a series that diverges.

   (e) True or false?: If $\lim(x_n) = 0$, then $\sum x_n$ must converge.

   (f) Prove that if $\sum x_n$ and $\sum y_n$ are both convergent series, then $\sum(x_n + y_n)$ is also a convergent series.

   (g) True or false?: If $\sum x_n$ and $\sum y_n$ both diverge, then $\sum(x_n + y_n)$ must diverge.

8. Section 4.1

   (a) Give the definition of "L is the limit of f at c." (Note: this is also written $\lim_{x\to c} f(x) = L$.

   (b) From the definition of limit, prove that for all real numbers c, $\lim_{x\to c} x^2 = c^2$.

   (c) State the Sequential Criterion for Limits.

   (d) Use the Sequential Criterion for Limits to prove that for all real numbers c, the function

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

   has no limit at c. (note: you will also need to use the Density Theorems 2.4.8 and 2.4.9.

9. Section 4.2

   (a) Prove that if $\lim_{x\to c} f(x) = L$ and if $\lim_{x\to c} g(x) = M$, then $\lim_{x\to c}(f(x) + g(x)) = L + M$.

   (b) State the Squeeze Theorem for limits of functions.