

Math 344, Section X1, Fall 2001
Review Problems for Chapters 1 and 2 - Solutions

Note: This is just a sampling of problems. It does not include everything.

1. Section 1.1

- (a) Give the definitions of “ $f : A \rightarrow B$ is injective” and “ $f : A \rightarrow B$ is surjective.”

Solution: See p. 8.

- (b) Give an example of a function which is injective and surjective, a function which is injective but not surjective, a function which is surjective but not injective, and a function which is neither injective nor surjective.

Solution: The function $f(x) = x$, from \mathbf{R} to \mathbf{R} is both injective and surjective. The function $g(x) = 2x$, from \mathbf{R} to \mathbf{R} , is injective but not surjective. The function $h(x) = x^2$, from \mathbf{R} to $\{y \in \mathbf{R} : y \geq 0\}$, is surjective but not injective. The function $k(x) = x^2$ from \mathbf{R} to \mathbf{R} is neither injective nor surjective.

- (c) Prove that if $f : A \rightarrow B$ and $g : B \rightarrow C$ and if $g \circ f$ is injective, then f must be injective.

Solution: I'll use the definition on p.8. Suppose that $x_1, x_2 \in A$ and $x_1 \neq x_2$. Then, since f is injective, $f(x_1) \neq f(x_2)$. Since g is also injective, $g(f(x_1)) \neq g(f(x_2))$. Since $g \circ f(x) = g(f(x))$, this shows that $f \circ g$ is injective.

2. Section 1.3

- (a) Give the definitions of “denumerable set”, “countable set” and “uncountable set”.

Solution: See p. 18.

- (b) Give an example of each type of set from part (a).

Solution: The natural numbers, the integers, the rational numbers are all examples of denumerable sets. Any denumerable set is also countable, and any finite set, for example $\{1, 3, 6\}$ is countable. The set of real numbers is uncountable.

- (c) Prove that the set of all vectors $\mathbf{v} = (v_1, v_2)$, where v_1 and v_2 are integers, is countable.

Solution: The set of integers is countable, so we may list the integers: a_1, a_2, a_3, \dots . The set of all vectors \mathbf{v} may be arranged in an array:

$$\begin{array}{lll} (a_1, a_1) & (a_1, a_2) & (a_1, a_3) \dots \\ (a_2, a_1) & (a_2, a_2) & (a_2, a_3) \dots \\ (a_3, a_1) & (a_3, a_2) & (a_3, a_3) \dots \\ \cdot & & \\ \cdot & & \\ \cdot & & \end{array}$$

We can then use a diagonal procedure to list all the vectors:

$$(a_1, a_1), (a_1, a_2), (a_2, a_1), (a_1, a_3), (a_2, a_2), (a_3, a_1), \dots,$$

similar to what is shown on p. 19. This proves that the set is denumerable, therefore countable.

3. Section 2.2

- (a) Give the definition of $|a|$.

Solution: See p. 31.

- (b) Prove that for all $a, b \in \mathbf{R}$, $|a + b| \leq |a| + |b|$.

Solution: See p. 31, Theorem 2.2.3.

- (c) Give an example where $|a + b| < |a| + |b|$. Give an example where $|a + b| = |a| + |b|$.

Solution: Let $a = -1$ and $b = 1$. Then $|a + b| = 0$ and $|a| + |b| = 2$, so $|a + b| < |a| + |b|$. Let $a = 1$ and $b = 1$. Then $|a + b| = 2 = |a| + |b|$.

4. Section 2.3

- (a) Give the definitions of upper bound of S , lower bound of S , $\sup S$, $\inf S$.

Solution: See pp. 35-36.

- (b) Give the statement of the Completeness Property of \mathbf{R} .

Solution: See p. 37.

- (c) Give an example of a set which has an upper bound but no lower bound.

Solution: The interval $(-\infty, 0]$.

- (d) Give an example of a bounded set S such that $\sup S$ is not an element of S .

Solution: The open interval $(0, 1)$ has 1 as its supremum, but 1 is not an element of the set.

- (e) Prove that if $A \subseteq B$ and A and B are both bounded, then $\inf A \geq \inf B$.

Solution: Let $a \in A$. Since $A \subseteq B$, $a \in B$. Since $\inf B$ is a lower bound for B , we have $a \geq \inf B$. This is true for all $a \in A$, so $\inf B$ is a lower bound for A . Since $\inf A$ is the *greatest* lower bound of A , $\inf A \geq \inf B$.

5. Section 2.4

- (a) Define “rational number” and “irrational number”.

Solution: See p. 24.

- (b) Prove that if x and y are both rational, then $x + y$ is rational.

Solution: If x and y are rational, then there exist integers a, b, c, d , with $b \neq 0$, $d \neq 0$, such that $x = a/b$ and $y = c/d$. Then $x + y = (ad + bc)/(bd)$. $ad + bc$ and bd are both integers and, since $b \neq 0$ and $d \neq 0$, we have $bd \neq 0$. Therefore $x + y$ is rational.

- (c) Prove that if x is rational and $x \neq 0$ and y is irrational, then xy is irrational.

Solution: The proof is by contradiction. Since x is rational and nonzero, there exist integers a and b , $b \neq 0$ and $a \neq 0$, such that $x = a/b$. Suppose that xy is rational. Then there exist integers c and d , $d \neq 0$, with $xy = c/d$. Multiply both sides by $1/x = b/a$ to get $y = (bc)/(ad)$. Since bc and ad are integers

with $ad \neq 0$, this contradicts the assumption that y is irrational. Therefore xy must not be rational; it is irrational.

- (d) Prove that if $x < y$ are two real numbers, then there is an irrational number z such that $x < z < y$ (i.e. fill in the details of the proof of Corollary 2.4.9.)

Solution: (refer to pp. 42-43) If $x < y$ then $x/\sqrt{2} < y/\sqrt{2}$. By the Density Theorem, there exists a rational number s such that $x/\sqrt{2} < s < y/\sqrt{2}$. If $s \neq 0$, let $r = s$. If $s = 0$, apply the Density Theorem again to find a rational number r such that $x/\sqrt{2} < 0 < r < y/\sqrt{2}$. Then, since $r \neq 0$ is rational and $1/\sqrt{2}$ is irrational, $r/\sqrt{2}$ is irrational (see the previous review problem for justification) and we have $x < r/\sqrt{2} < y$.