

## Math 344, Section X1, Fall 2001

### Review Problems for Chapters 3 and 4 - Solutions

Note: This is just a sampling of problems. It does not include everything. The test will be on Monday, Oct. 22 and will cover Sections 3.1-3.5, 3.7, 4.1-4.2.

#### 1. Section 3.1

- (a) Give the definition of “the sequence  $(x_n)$  converges to  $x$ .”

**Solution:** See p. 54, Definition 3.1.3.

- (b) Prove that if  $(x_n)$  converges to  $x'$  and if the same sequence  $(x_n)$  also converges to  $x''$ , then  $x' = x''$ .

**Solution:** See p. 54, Theorem 3.1.4.

- (c) Give an example of a sequence which converges to 2. Give an example of a sequence that diverges.

**Solution:** There are many possible answers.  $(2 + \frac{1}{n})$  converges to 2. The constant sequence  $x_n = 2$  for all  $n$  also converges to 2. A few examples of divergent sequences are  $(n)$ ,  $((-1)^n)$ ,  $(n^2)$ .

- (d) Prove that  $((-1)^n)$  diverges (does not converge to any  $x$ ).

**Solution:** Let  $c$  be a real number. We will show that  $((-1)^n)$  does not converge to  $c$ . Let  $\epsilon = \frac{1}{2}$ . Let  $K$  be any natural number. No matter what  $K$  is, it cannot be true that  $|(-1)^n - c| < \epsilon$  for all  $n \geq K$ , for if this were true, then let  $n_1 \geq K$  be odd and let  $n_2 \geq K$  be even. Then  $2 = |1 - (-1)| = |1 - c + c - (-1)| \leq |1 - c| + |c - (-1)| = |1 - c| + |-1 - c| = |(-1)^{n_2} - c| + |(-1)^{n_1} - c| < \epsilon + \epsilon = \frac{1}{2} + \frac{1}{2} = 1$ . This would imply  $2 < 1$ . So either  $|(-1)^{n_2} - c| \geq \epsilon$  or  $|(-1)^{n_1} - c| \geq \epsilon$ . Therefore  $\epsilon = \frac{1}{2}$  is an  $\epsilon > 0$  for which there is no  $K$  which “works”, so the  $((-1)^n)$  does not converge to  $c$ , for all real numbers  $c$ . Therefore the sequence diverges.

An easier proof can be given using Theorem 3.4.2. If  $((-1)^n)$  converges to a number  $c$ , then any subsequence converges to the same number  $c$ . The subsequence  $(1, 1, 1, 1, 1, 1, \dots)$  converges to 1 and the subsequence  $(-1, -1, -1, -1, \dots)$  converges to  $-1$ . Since  $c$  cannot equal both 1 and  $-1$ , the sequence cannot converge to  $c$ ; it diverges. (Note: you could apply the Divergence Criteria 3.4.5 (i), but since this follows directly from 3.4.2, why remember an additional theorem? Also, 3.4.5 (ii) follows directly from 3.2.2, so just remember 3.2.2 - a convergent sequence must be bounded.)

#### 2. Section 3.2

- (a) Give the definition of “bounded sequence”.

**Solution:** See p. 60.

- (b) Prove that if  $(x_n)$  converges, then  $(x_n)$  is bounded.

**Solution:** See pages 60-61, Theorem 3.2.2.

- (c) Give an example of a bounded sequence which diverges.  
**Solution:**  $((-1)^n)$  is bounded and diverges. Whatever example you give will not be monotone, because of the Monotone Convergence Theorem.
- (d) Prove that if  $(x_n)$  converges to  $x$  and if  $(y_n)$  converges to  $y$ , then  $(x_n + y_n)$  converges to  $x + y$ .  
**Solution:** See the proof of Theorem 3.2.3 on pages 61-62.

### 3. Section 3.3

- (a) Give the definitions of “increasing sequence,” “decreasing sequence,” “monotone sequence.”  
**Solution:** See page 69.
- (b) Give an example of a monotone sequence which diverges. Give an example of a sequence which converges but which is not monotone.  
**Solution:** The sequence  $(n)$  is monotone increasing and diverges. The sequence  $((-1)^n/n)$  converges to 0 and is not monotone.
- (c) State the Monotone Convergence Theorem.  
**Solution:** A bounded, monotone sequence (of real numbers) must converge. (See p. 69 for a little more detail.)

### 4. Section 3.4

- (a) Give the definition of “subsequence.”  
**Solution:** See page 75.
- (b) True or false?: If  $(x_{n_k})$  is a subsequence of  $(x_n)$  and if  $(x_{n_k})$  converges to  $x$ , then  $(x_n)$  must converge to  $x$ .  
**Solution:** False. For example, in the sequence  $(1, 0, 1, 0, 1, 0, \dots)$ , the subsequence  $(1, 1, 1, \dots)$  converges to 1, but the original sequence does not converge.
- (c) True or false?: If  $(x_{n_k})$  is a subsequence of  $(x_n)$  and if  $(x_n)$  converges to  $x$ , then  $(x_{n_k})$  must converge to  $x$ .  
**Solution:** True, by Theorem 3.4.2 on page 76.
- (d) True or false?: If  $(x_n)$  diverges, then  $(x_n)$  must be unbounded.  
**Solution:** False.  $((-1)^n)$  diverges and is bounded.
- (e) True or false?: If  $(x_n)$  is unbounded, then  $(x_n)$  diverges.  
**Solution:** True, because if  $(x_n)$  converges, then by Theorem 3.2.2, it must be bounded.
- (f) True or false?: Every sequence has a convergent subsequence.  
**Solution:** False. For example, the sequence  $(n)$  has no convergent subsequence.
- (g) State the Bolzano-Weierstrass Theorem.  
**Solution:** Every bounded sequence has a convergent subsequence. (See page 78).

5. General question for Chapter 3: List as many ways as you can think of to show that a sequence of real numbers converges. List as many ways as you can think of to show that a sequence of real numbers diverges.

6. Section 3.5

(a) Give the definition of Cauchy sequence.

**Solution:** See page 81.

(b) Prove that  $(\frac{1}{n})$  is a Cauchy sequence.

**Solution:** See page 81, Example 3.5.2(a).

(c) Prove that a convergent sequence must be Cauchy.

**Solution:** See page 81, proof of Lemma 3.5.3.

(d) True or false?: A Cauchy sequence (of real numbers) must converge.

**Solution:** True. The Cauchy Convergence Theorem (p. 82) says that a sequence of real numbers converges if and only if it is Cauchy.

7. Section 3.7

(a) What is the difference between a sequence and a series?

**Solution:** In informal language, a sequence is a list of numbers, whereas a series is an infinite sum - the numbers are added together.

(b) Give the definition of partial sums.

**Solution:** See page 89.

(c) Give an example of a series that converges.

**Solution:** One example is a geometric series  $\sum r^n$ , with  $|r| < 1$ , such as  $\sum (1/2)^n$ .

(d) Give an example of a series that diverges.

**Solution:**  $\sum n$  diverges (its sequence of partial sums is unbounded.)  $\sum (-1)^n$  also diverges - see Example 3.7.2 (b) on page 90.

(e) True or false?: If  $\lim(x_n) = 0$ , then  $\sum x_n$  must converge.

**Solution:** This is false. For example, the harmonic series  $\sum \frac{1}{n}$  diverges but has  $\lim(\frac{1}{n}) = 0$ . The correct statement is that if  $\sum x_n$  converges, then  $\lim(x_n) = 0$  - See the nth Term Test on page 91.

(f) Prove that if  $\sum x_n$  and  $\sum y_n$  are both convergent series, then  $\sum(x_n + y_n)$  is also a convergent series.

**Solution:** Let  $(s_k)$  be the sequence of partial sums of  $\sum x_n$  and suppose that  $\lim(s_k) = s$  (because  $\sum x_n$  converges). Let  $(t_k)$  be the sequence of partial sums of  $\sum y_n$  and suppose that  $\lim(t_k) = t$  (because  $\sum y_n$  converges). Then the sequence of partial sums of  $\sum x_n + y_n$  is  $(s_k + t_k)$ . By Theorem 3.2.3,  $(s_k + t_k)$  converges to  $s + t$ , which means, by definition, that  $\sum x_n + y_n$  is convergent.

(g) True or false?: If  $\sum x_n$  and  $\sum y_n$  both diverge, then  $\sum(x_n + y_n)$  must diverge.

**Solution:** False. For example,  $x_n = n$ ,  $y_n = -n$ .

## 8. Section 4.1

- (a) Give the definition of “ $L$  is the limit of  $f$  at  $c$ .” (Note: this is also written  $\lim_{x \rightarrow c} f(x) = L$ .)

**Solution:** See page 98.

- (b) From the definition of limit, prove that for all real numbers  $c$ ,  $\lim_{x \rightarrow c} x^2 = c^2$ .

**Solution:** See Example (c) on p. 100.

- (c) State the Sequential Criterion for Limits.

**Solution:** See Theorem 4.1.8 on page 101.

- (d) Use the Sequential Criterion for Limits to prove that for all real numbers  $c$ , the function

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

has no limit at  $c$ . (note: you will also need to use the Density Theorems 2.4.8 and 2.4.9.)

**Solution:** Given a number  $c$ , for each natural number  $n$ , let  $x_n$  be a rational number between  $c$  and  $c + \frac{1}{n}$ . The Density Theorem 2.4.8 ensures that such  $x_n$  exists. Then  $(x_n)$  is a sequence which converges to  $c$ . The sequence  $(f(x_n))$  is the constant sequence (1) since each  $x_n$  is rational, so  $(f(x_n))$  converges to 1.

For each natural number  $n$ , let  $y_n$  be an irrational number between  $c$  and  $c + \frac{1}{n}$ . The Density Theorem 2.4.9 ensure that such  $y_n$  exists. Then  $(y_n)$  is a sequence which converges to  $c$ . The sequence  $(f(y_n))$  is the constant sequence (0), since each  $y_n$  is irrational, so  $(f(y_n))$  converges to 0.

By the Sequential Criterion for Limits, if  $f(x)$  has a limit at  $c$ , then the sequences  $(f(x_n))$  and  $(f(y_n))$  would both converge to this limit. Since they converge to two different numbers, we conclude that  $f(x)$  has no limit at  $c$ .

## 9. Section 4.2

- (a) Prove that if  $\lim_{x \rightarrow c} f(x) = L$  and if  $\lim_{x \rightarrow c} g(x) = M$ , then  $\lim_{x \rightarrow c} (f(x) + g(x)) = L + M$ .

**Solution:** Let  $A$  be the domain of  $f$  and  $g$ . Since the limits of  $f$  and  $g$  exist at  $c$ , it must be that  $c$  is a cluster point of  $A$ . Let  $(x_n)$  be a sequence in  $A$  which has no terms equal to  $c$  but which converges to  $c$ . By the sequential criterion,  $\lim(f(x_n)) = L$  and  $\lim(g(x_n)) = M$ . By Theorem 3.2.3 on limits of sequences,  $\lim(f(x_n) + g(x_n)) = L + M$ . Since this is true for any sequence  $(x_n)$  meeting the conditions above, we conclude from the Sequential Criterion that  $\lim_{x \rightarrow c} (f(x) + g(x)) = L + M$ .

- (b) State the Squeeze Theorem for limits of functions.

**Solution:** See Theorem 4.2.7 on page 108.