

Math 344, Section X1, Fall 2001

Review Problems for Chapters 6 and 7 and Section 5.4

Note: This is just a sampling of problems. It does not include everything.

1. Section 6.1

- (a) Give the definitions of “ f is differentiable at c ” and “ f is differentiable on the set I ”.

Solution: See 6.1.1 on page 158. f is differentiable on the set I if f is differentiable at c for every $c \in I$.

- (b) Give an example of a function which is differentiable on \mathbf{R} , a function which is continuous on \mathbf{R} but which is not differentiable at 0, a function which is defined on \mathbf{R} but which is not differentiable at any point.

Solution: $f(x) = x$ is differentiable on \mathbf{R} , $f(x) = |x|$ is continuous on \mathbf{R} but is not differentiable at 0, and $f(x) = 0$ for x rational, $= 1$ for x irrational is not differentiable at any point (because it is not continuous at any point).

- (c) Prove that if f and g are both differentiable at c , then fg is differentiable at c and $(fg)' = f'g + fg'$.

Solution: See page 160, part (c).

- (d) State the Chain Rule, including all hypotheses.

Solution: See 6.1.6 on page 162.

2. Section 6.2

- (a) State the Interior Extremum Theorem and give a proof in the case where f has a relative minimum at c .

Solution: See 6.2.1 on page 168. For the proof, suppose f has a relative minimum at c and c is an interior point of the interval I . If $f'(c) > 0$, then there is an open interval $V \subset I$ around c such that $f(x) - f(c)/(x - c) > 0$ for $x \in V$, $x \neq c$ (see Theorem 4.2.9 and the definition of $f'(c)$). If $x \in V$ and $x < c$, then

$$f(x) - f(c) = (x - c) \cdot \frac{f(x) - f(c)}{x - c} < 0.$$

This contradicts the hypothesis that f has a relative minimum at c . If $f'(c) < 0$, then there is an open interval $V \subset I$ around c such that $f(x) - f(c)/(x - c) < 0$ for $x \in V$, $x \neq c$ (see Theorem 4.2.9 and the definition of $f'(c)$). If $x \in V$ and $x > c$, then

$$f(x) - f(c) = (x - c) \cdot \frac{f(x) - f(c)}{x - c} < 0.$$

This again contradicts the hypothesis that f has a relative minimum at c . Since we are assuming that $f'(c)$ exists and since $f'(c) > 0$ and $f'(c) < 0$ both lead to contradictions, it must be that $f'(c) = 0$.

- (b) Is the Interior Extremum Theorem true if c is an endpoint of I ? Give a counterexample or a proof.

Solution: No, the theorem would be false in that case. For example, let $I = [0, 1]$ and $f(x) = x$. Then f has a relative maximum (on I) at $c = 1$, but $f'(c) \neq 0$.

- (c) State the Mean Value Theorem.

Solution: See 6.2.4 on page 169.

- (d) Is the Mean Value Theorem true if the hypothesis of continuity on $[a, b]$ is dropped? Prove or give a counterexample.

Solution: It is false. For example, let $I = [0, 1]$ and let

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1. \end{cases}$$

Then $f'(c) = 0$ for all $c \in (0, 1)$, but $f(1) - f(0) = 1$, so there is no c such that $f(1) - f(0) = f'(c)(1 - 0)$.

- (e) Is the Mean Value Theorem true if the hypothesis of differentiability on (a, b) is dropped? Prove or give a counterexample.

Solution: It is false. For example, let $I = [-1, 1]$ and let $f(x) = |x|$. Then $f'(c) = 1$ for $c > 0$ and $f'(c) = -1$ for $c < 0$ and $f'(0)$ is undefined, but $f(1) - f(-1) = 0$, so there is no c such that $f(1) - f(-1) = f'(c)(1 - (-1))$.

3. Section 7.1

- (a) Give the definition of Riemann integrable.

Solution: See 7.1.1 on page 196.

- (b) Prove, directly from the definition, that a constant function is Riemann integrable on the interval $[a, b]$.

Solution: See Example 7.1.3(a) on page 197.

- (c) Is it true that a bounded function must be integrable on $[a, b]$? Give some examples.

Solution: No. For example, $f(x) = 0$ for x rational and $= 1$ for x irrational is bounded on any closed interval but is not Riemann integrable.

- (d) Prove (from the definition of Riemann integral), that if f is Riemann integrable on $[a, b]$, then $\int_a^b 2f = 2 \int_a^b f$. Let $\epsilon > 0$. Since f is Riemann integrable on $[a, b]$, there exists δ such that for all tagged partitions P with $\|P\| < \delta$, we have $|S(f; P) - \int_a^b f| < \epsilon/2$. Observe that $S(2f; P) = 2S(f; P)$; this follows directly from the definition of Riemann sum. Therefore, for all tagged partitions P with $\|P\| < \delta$, we have $|S(2f; P) - 2 \int_a^b f| = |2S(f; P) - 2 \int_a^b f| = 2|S(f; P) - \int_a^b f| < 2 \cdot \epsilon/2 = \epsilon$. By definition of Riemann integral, $\int_a^b 2f = 2 \int_a^b f$.

4. Section 7.2

- (a) State the Cauchy Criterion and give an example of how it can be used.
Solution: For the statement, see 7.2.1 on page 203. 7.2.2 gives several examples of its use - see also the homework problems from this section.
- (b) State the Squeeze Theorem and give an example of how it can be used.
Solution: See 7.2.3 on page 204. The Squeeze Theorem is used to prove that continuous functions are integrable on $[a, b]$ - see the proof of Theorem 7.2.6 on page 205.
- (c) True or false (give examples or cite Theorems from the book to support your answer.):
- i. Every continuous function on $[a, b]$ is Riemann integrable.
Solution: True. Theorem 7.2.6 on page 205.
 - ii. Every step function on $[a, b]$ is Riemann integrable.
Solution: True. Theorem 7.2.5 on page 205.
 - iii. Every bounded function on $[a, b]$ is Riemann integrable.
Solution: False. For example, $f(x) = 1$ if x is rational, $= 0$ if x is irrational is bounded on any $[a, b]$ but is not Riemann integrable (See Example 7.2.2(b) on page 204).
 - iv. Every monotone function on $[a, b]$ is Riemann integrable.
Solution: True. Theorem 7.2.7 on page 206.
 - v. Every function which is Riemann integrable on $[a, b]$ must be continuous.
Solution: False. For example, a step function is integrable by Theorem 7.2.5, but need not be continuous.
 - vi. Every function which is Riemann integrable on $[a, b]$ must be a step function.
Solution: False. By Theorem 7.2.6, every continuous function must be integrable, and there are many continuous functions (such as $f(x) = x$) which are not step functions.
 - vii. Every function which is Riemann integrable on $[a, b]$ must be bounded.
Solution: True. Theorem 7.1.5 on page 200.
 - viii. Every function which is Riemann integrable on $[a, b]$ must be monotone.
Solution: False. For example, $f(x) = x^2$ is integrable on $[-1, 1]$ because it is continuous (7.2.6), but it is not monotone on $[-1, 1]$.

5. Section 7.3

- (a) State the two forms of the Fundamental Theorem of Calculus. Be sure to include all hypotheses.
Solution: See 7.3.1 on page 210 and 7.3.5 on page 212.
- (b) Find $F'(x)$ when F is defined on $[0, 1]$ as follows (verify that the hypotheses of whatever theorem you use are satisfied):

$$F(x) = \int_{x^3}^x \sin(t^2 + 1) dt.$$

Solution: We will use the 2nd Form of the Fundamental Theorem of Calculus, 7.3.5. Let $f(t) = \sin(t^2 + 1)$. Since f is the sine function composed with a polynomial, f is continuous everywhere, so f is integrable on every closed bounded interval. Let

$$G(u) = \int_0^u \sin(t^2 + 1) dt.$$

Then, by 7.3.5, $G'(u) = \sin(u^2 + 1)$. G is continuous by 7.3.4. Let $u = \phi(x) = x^3$; ϕ has a continuous derivative everywhere. Therefore we can apply the Chain Rule, the Additivity Theorem and the 2nd Form of the Fundamental Theorem of Calculus to get:

$$F(x) = - \int_0^{x^3} \sin(t^2 + 1) dt + \int_0^x \sin(t^2 + 1) dt.$$

$$\begin{aligned} F'(x) &= -G'(u)\phi'(x) + \sin(x^2 + 1) \\ &= -\sin(u^2 + 1) \cdot 3x^2 + \sin(x^2 + 1) = -3x^2 \sin(x^6 + 1) + \sin(x^2 + 1). \end{aligned}$$

6. Section 5.4

- (a) Give the definition of “ f is uniformly continuous on $A \subset \mathbf{R}$.”

Solution: See 5.4.1 on page 137.

- (b) Give an example of a function which **is** uniformly continuous on the open interval $(0, 1)$.

Solution: $f(x) = x$ is uniformly continuous on $(0, 1)$. To see this, let $\epsilon > 0$ and let $\delta = \epsilon$. If $|x - u| < \delta$, Then $|f(x) - f(u)| = |x - u| < \delta = \epsilon$.

- (c) Give an example of a function which **is not** uniformly continuous on the open interval $(0, 1)$.

Solution: The function $f(x) = 1/x$ is not uniformly continuous on $(0, 1)$. To see this, let $\epsilon_0 = 1$. Let δ be a positive number less than 1. Choose $x \in (0, 1)$ small enough so that

$$\frac{\delta/2}{(x + \delta/2)^2} \geq 1.$$

Algebra shows that this inequality is equivalent to $x \leq \sqrt{\delta/2} - \delta/2$. Choose $u = x + \delta/2$. Then $|x - u| = \delta/2 < \delta$ and

$$|f(x) - f(u)| = \left| \frac{1}{x} - \frac{1}{u} \right| = \left| \frac{u - x}{xu} \right| = \frac{\delta/2}{x(x + \delta/2)} \geq \frac{\delta/2}{(x + \delta/2)^2} \geq 1.$$

Since such an x can be found for any $\delta < 1$, f is not uniformly continuous on $(0, 1)$.

- (d) Prove that if (x_n) is a Cauchy sequence in a set $A \subset \mathbf{R}$ and if f is uniformly continuous on A , then $(f(x_n))$ is a Cauchy sequence.

Solution: See the proof of Theorem 5.4.7 on page 140.