

## Answers to Practice on Topological Properties - 9/29/00

1. Closed, bounded, compact. Interior =  $\{x : 1 < |x| < 2\}$ , Closure = the set itself, Boundary =  $\{x : |x| = 1\} \cup \{x : |x| = 2\}$ .
2. Closed. Interior =  $\{x : x_1 > 0\}$ , Closure = the set itself, Boundary =  $\{x : x + 1 = 1\}$ .
3. Closed. Interior = empty set, Closure = the set itself, Boundary = the set itself.
4. Has none of the four listed properties. Interior = empty set, Closure = all of  $\mathbf{R}^2$ , Boundary = all of  $\mathbf{R}^2$ .
5. Closed. Interior = empty set. Closure = the set itself. Boundary = the set itself.
6. Open, closed, bounded, compact. The interior, closure, and boundary are all the empty set.
7. Closed. Interior = empty set, Closure = the set itself, Boundary = the set itself.
8. Closed, bounded, compact. Interior = empty set. Closure = the set itself. Boundary = the set itself.
9. Closed. Interior = the empty set. Closure = the set itself. Boundary = the set itself.
10. Open, bounded. Interior = the set itself. Closure =  $\{f \in S : |f(x)| \leq 1 \text{ for all } x \in [0, 1]\}$ . Boundary =  $\{f \in S : |f(x)| = 1 \text{ for all } x \in [0, 1]\}$ .
11. The set of polynomials has none of the four listed properties. Interior = empty set. Closure and boundary = the set of all continuous functions on  $[0, 1]$ . (The closure and boundary are not obvious - Weierstrass's Approximation Theorem in Section 27 of the book gives us this information.)
12. Open. Interior = the set itself. Closure = the set of real numbers. Boundary = the set of integers.
13. Closed. Interior =  $\cup_{n \in \mathbf{Z}} (2n, 2n + 1)$ . Closure = the set itself. Boundary = the set of integers.
14. False. For example, let  $E_n = [1/n, 1]$  in  $\mathbf{R}$ .  $E_n$  is closed but  $\cup_n E_n = (0, 1]$ , which is not closed.
15. True. Let  $E$  be compact. Fix a point  $s_0 \in S$ . For  $n \in \mathbf{N}$ , let  $U_n = \{s \in S : d(s, s_0) < n\}$ . This is an open cover of  $E$  (in fact of  $S$ ), so it has a finite subcover of  $E$ . Since the sets  $U_n$  are nested, this means that for some  $n$ ,  $E \subset U_n$ , so for all  $s \in E$ ,  $d(s, s_0) < n$ . Now let  $s, t \in E$ . Then  $d(s, t) \leq d(s, s_0) + d(t, s_0) < 2n$ , so the diameter of  $E$  is at most  $2n$ ; in particular, the diameter is finite.

16. False. Some open covers will have a finite subcover. For example, let the cover consist of the single set  $S$  (the whole metric space). Then the single set  $S$  is a finite subcover of that cover. (What we can say is that if  $E$  is not compact, then there is *some* open cover which has no finite subcover.)
17. False. For example, the interval  $(0, 1]$  in  $\mathbf{R}$  is neither open nor closed.
18. True, because the definition of boundary of  $E$  is  $E^- \setminus E^\circ$ .
19. False. For example,  $\{1/n : n \in \mathbf{N}\}$  in  $\mathbf{R}$  has no interior and is not closed.
20. True. By Proposition 13.9d, the boundary is the intersection of two closed sets, so it is closed.