

Name: _____

Math 385, Section D1 - Final Exam -December 15, 2005

Time: 3 hours. You may not use any books or notes or calculator. There are 200 points possible. To get full credit, you must show your work.

The following formulas are given:

$$x_{\text{sp}}(t) = \sum_{n=1}^{\infty} \frac{B_n \sin(\omega_n t - \alpha_n)}{\sqrt{(k - m\omega_n^2)^2 + (c\omega_n)^2}}$$

$$u(x, t) = \sum_{n=1}^{\infty} b_n \exp(-n^2 \pi^2 kt / L^2) \sin \frac{n\pi x}{L}$$

$$u(x, t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \exp(-n^2 \pi^2 kt / L^2) \cos \frac{n\pi x}{L}$$

1. (15 points each part) If no initial conditions are given, find the general solution (i.e. all solutions) to the differential equation. If initial conditions are given, find the solution satisfying those initial conditions.

(a) (15 points) $y' - y \sin x = 0$, $y(0) = 1$.

(b) (15 points)

$$3x^2 + 2y^2 + (4xy + 6y^2)\frac{dy}{dx} = 0.$$

(c) (15 points)

$$x \frac{dy}{dx} + 6y = 3xy^{4/3}.$$

(d) (15 points) $y'' + 3y' - 10y = 0$, $y(0) = 7$, $y'(0) = 7$.

(e) (15 points) $y'' - y = xe^x$.

(f) (15 points) $x'' + 9x = F(t)$, where $F(t)$ is the odd function of period 2π with $F(t) = 1$ on $0 < t < \pi$.

2. (15 points) The equation $\frac{dx}{dt} = x(10 - x)$ models the growth of an insect population, with $x(t)$ = the number of insects in thousands after t days. Draw the slope field for this equation. What will happen to the insect population if you begin with 20,000 insects? Explain your answer.

3. (15 points) A body with a mass of 1 kg is attached to a spring with a spring constant $k = 5$ and to a dashpot which provides resistance with $c = 2$. (*Note*: all units of measurement correspond - no need to adjust units). There is no forcing function. Is this system overdamped, underdamped, or critically damped. Explain how you arrived at your answer.

4. (20 points) A copper rod 50 cm. long with insulated lateral surface has initial temperature $u(x, 0) = 2x$, and its two ends are insulated beginning at time $t = 0$. Find $u(x, t)$.

5. (9 points) Consider the initial value problem $y' = x - y$, $y(0) = 2$. Using a step size of $h = 0.5$, do two steps of Euler's method to approximate the value of the solution at $x = 1$. (*Note:* It is possible to solve this equation exactly, but the point here is to show that you know the steps of Euler's method.)

6. (3 points each part) Answer *True* or *False* for each part. No explanation is needed and this question has no partial credit, just right or wrong.

(a) The function $y(x) = x$ is a solution of the differential equation

$$\frac{dy}{dx} = \sin^2(x^2) + \cos^2(xy).$$

(b) The solution to $y'' + 4y = \sin 16x$ will exhibit pure resonance.

(c) The functions $y_1(x) = x$ and $y_2(x) = x^2$ are linearly independent on the interval $(-1, 1)$.

(d) If $f(t)$ is periodic and piecewise smooth, then its Fourier series converges to the value $\frac{1}{2}[f(t^+) + f(t^-)]$ at each point where f is discontinuous.

(e) Given any function $f(x, y)$ and any initial condition $y(a) = b$, the differential equation

$$\frac{dy}{dx} = f(x, y)$$

has a unique solution with $y(a) = b$.

(f) For the equation $y'' = y(y - 1)$, the critical point $y = 1$ is stable.

(g) If a function $f(t)$ is odd, then its Fourier series contains only the constant term and cosine terms. (We're assuming that $f(t)$ meets all the conditions to have a convergent Fourier series.)