Name: Solutions

Math 385, Section D1 - Test #3 -December 9, 2005

Time: 55 minutes. You may not use any books or notes or calculator. There are 100 points possible. To get full credit, you must show your work.

The following formulas are given:

$$x_{\mathrm{sp}}(t) = \Sigma_{n=1}^{\infty} \frac{B_n \sin(\omega_n t - \alpha_n)}{\sqrt{(k - m\omega_n^2)^2 + (c\omega_n)^2}}$$

$$u(x,t) = \Sigma_{n=1}^{\infty} b_n \exp(-n^2\pi^2 kt/L^2) \sin\frac{n\pi x}{L}$$

$$u(x,t) = \frac{a_0}{2} + \Sigma_{n=1}^{\infty} a_n \exp(-n^2\pi^2 kt/L^2) \cos\frac{n\pi x}{L}$$

1. (a) (5 points) Give the definition of *even function*.

f is even if $f(-t) = f(t)$ for all t in the domain of f.

(b) (10 points) Suppose that f is an even function. Show that

$$\int_{-a}^{0} f(t)\,dt = \int_{0}^{a} f(t)\,dt.$$

Substitution: $u = -t$ $\quad dt = -du \quad$ when $t = -a$, $u = a$

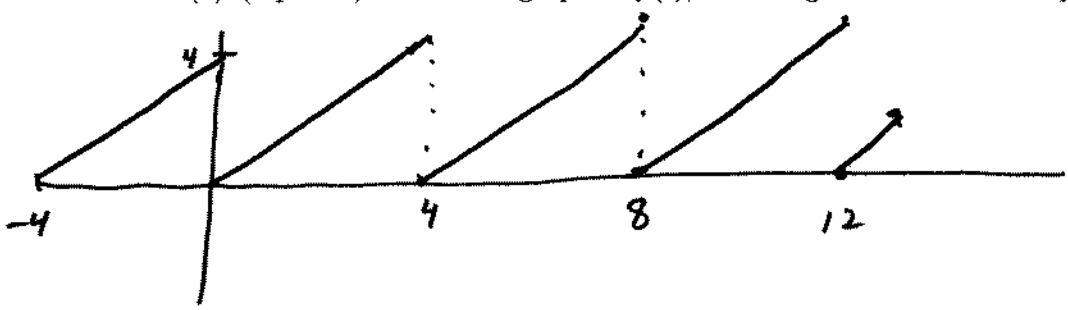
$$\int_{-a}^{0} f(t)dt = -\int_{a}^{0} f(-u)\,du = \int_{0}^{a} f(-u)\,du$$

$$= \int_{0}^{a} f(u)\,du \quad (\text{since } f \text{ is even})$$

$$= \int_{0}^{a} f(t)\,dt$$

2. Let $f(t)$ be a function of period 4, given by $f(t) = t$ for $0 < t < 4$.

(a) (6 points) Sketch the graph of $f(t)$, including at least three full periods.



(b) (10 points) Write the Fourier series for $f(t)$. *Important note:* you may give the Fourier coefficients a_n and b_n as definite integrals, without evaluating those integrals.

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{2} + b_n \sin\frac{n\pi t}{2}$$

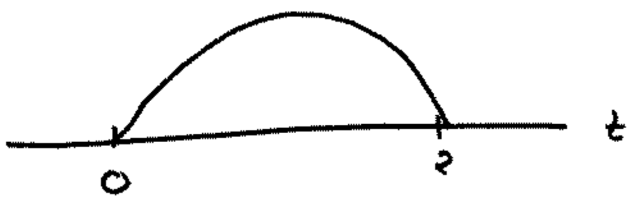
$$\text{where} \quad a_0 = \frac{1}{2}\int_0^4 t \,\cancel{\cos}\, dt$$

$$a_n = \cancel{2}\cdot\frac{1}{2}\int_0^4 t \cos\frac{n\pi t}{2}\, dt$$

$$b_n = \cancel{2}\cdot\frac{1}{2}\int_0^4 t \sin\frac{n\pi t}{2}\, dt$$

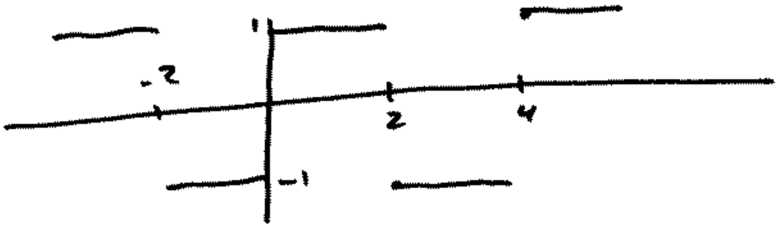
3. (15 points) Find the formal Fourier series solution of the endpoint value problem

$$x'' - 4x = 1, \quad x(0) = x(2) = 0.$$



$\sin \frac{n\pi t}{2}$

Extend 1 as odd, period 4



$f(t)$ $\quad L=2$

Fourier coeffs: $a_n = 0$ since f is ~~even~~ odd

$$b_n = \frac{1}{2}\int_{-2}^{2} f(t)\sin\frac{n\pi t}{2}\,dt = 2\cdot\frac{1}{2}\int_0^2 1\cdot\sin\frac{n\pi t}{2}\,dt$$

$$= -\frac{2}{n\pi}\cos\frac{n\pi t}{2}\Big|_0^2 = -\frac{2}{n\pi}\left(\cos n\pi - 1\right) = \frac{4}{n\pi} \text{ if } n \text{ odd}$$

(0 if n even)

$$x = \sum_{n \text{ odd}} \beta_n \sin\frac{n\pi t}{2} \qquad x'' = \sum_{n \text{ odd}} -\frac{n^2\pi^2}{4}\beta_n \sin\frac{n\pi t}{2}$$

$$x'' - 4x = \sum_{n \text{ odd}} \beta_n\left(-\frac{n^2\pi^2}{4} - 4\right)\sin\frac{n\pi t}{2} = f(t) = \sum_{n \text{ odd}} \frac{4}{n\pi}\sin\frac{n\pi t}{2}$$

$$\beta_n = \frac{4}{n\pi} \div \left(\frac{-n^2\pi^2}{4} - 4\right)$$

and $x(t) = \sum_{n \text{ odd}} \beta_n \sin\frac{n\pi t}{2}$

4. A rod 5 cm long is made of a material with "thermal diffusivity constant" $k = 0.1\ cm^2/s$ and has insulated lateral surfaces. It is heated to a temperature of $f(x) = \sin\frac{\pi x}{5}$ degrees and at time $t = 0$, its two ends are embedded in ice at 0 degrees.

(a) (10 points) Find the temperature $u(x,t)$ of the rod. *Important note:* You can find the Fourier series for the function $f(x)$ without computing any integrals - look at the form of $f(x)$.

0 ——— 5

$L = 5 \qquad k = 0.1$

0 endpts.

$f(x)$: $\sin\frac{\pi x}{5}$ so $B_1 = 1$, all other $B_n = 0$

$$\boxed{u(x,t) = \cancel{\sum_{n}} e^{\frac{-\pi^2(.1)t}{25}} \sin\frac{\pi x}{5}}$$

(b) (6 points) What differential equation does $u(x,t)$ satisfy? In other words, give the heat equation.

$$u_t = 0.1\, u_{xx}$$

(c) (6 points) What is the temperature of the midpoint of the rod after 10 minutes (600 seconds)?

$x = 5/2 \quad t = 600$

$$u\left(\tfrac{5}{2}, 600\right) = e^{\frac{-\pi^2(.1)600}{25}} \underbrace{\sin\tfrac{\pi}{2}}_{1} = e^{\frac{-\pi^2 60}{25}}$$

5. (12 points) Consider a mass-and-spring system with mass $m = 3$ and Hooke's constant $k = 12$. Suppose the system is under the influence of a periodic forcing function $F(t)$ which has Fourier series

$$F(t) = \Sigma_{n=1}^{\infty} \frac{3\pi}{n} \sin n\pi t.$$

Does pure resonance occur? Explain your answer.

$3x'' + 12x = 0$ (unforced)

$x'' + 4x = 0$

natural frequency $= 2$.

Can $n\pi = 2$? NO, since ~~n is~~ n is integer

2 is not one of the frequencies of $F(t)$, so pure resonance does not occur

6. (4 points each part) Answer *True* or *False* for each part. No explanation is needed and this question has no partial credit, just right or wrong.

(a) For any piecewise smooth function $f(t)$ which is defined for all t, the Fourier series is defined and converges.

F (must be periodic)

(b) Let $f(t)$ be the odd function of period 2 which is defined as $f(t) = 4$ for $0 < t < 1$. When $t = 1$, the Fourier series converges to 0.

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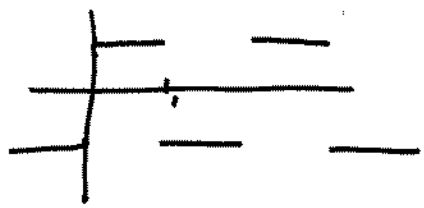

converges to $\frac{1}{2}(f(1^+) + f(1^-))$

$= \frac{1}{2}(4-4) = 0$

(c) If $f(t)$ is periodic with period 10, then $\int_0^{10} f(t)\,dt = \int_{-5}^{5} f(t)\,dt$.

T $\int_0^{10} f(t)\,dt = \int_a^{a+10} f(t)\,dt$ for any a.

(d) The function $f(x)$ which is defined as $f(x) = 1/x$ for $x > 0$ and $f(x) = 2$ for $x \le 0$ is piecewise continuous.

F false the discontinuity at $x=0$ is not a jump discontinuity.

(e) The method called "separation of variables" is used to find the steady periodic solution of an equation of the form $mx'' + cx' + kx = F(t)$, where $F(t)$ is periodic.

F It is used for the heat equation.