Section 3.1

① $y'' - y = 0.$

a. Verify $y_1 = e^x$ is a solution: $y_1'' = e^x$, so $y_1'' - y_1 = 0$. y_1 satisfies the given D.E.

b. Verify $y_2 = e^{-x}$ is a solution: $y_2' = -e^{-x}$ and $y_2'' = e^{-x}$, so $y_2'' - y_2 = 0$.

c. Let $y = c_1 e^x + c_2 e^{-x}$

then $y' = c_1 e^x - c_2 e^{-x}$

Plug in $y(0) = 0$, $y'(0) = 5$

$0 = c_1 + c_2$; $5 = c_1 - c_2$.

We get $c_1 = 5/2$ and $c_2 = -5/2$.

The particular solution is $y = \frac{5}{2} e^x - \frac{5}{2} e^{-x}$.

③ $y'' + 4y = 0.$

a. Verify $y_1 = \cos 2x$ is a solution: $y_1' = -2\sin 2x$ and $y_1'' = -4\cos 2x$ so $y_1'' + 4y_1 = 0$.

b. Verify $y_2 = \sin 2x$ is a solution: $y_2' = 2\cos 2x$ and $y_2'' = -4\sin 2x$ so $y_2'' + 4y_2 = 0$

Section 3.1

③ cont'd

c. Let $y = c_1 \cos 2x + c_2 \sin 2x$

$y' = -2c_1 \sin 2x + 2c_2 \cos 2x$

Plug in $y(0)=3$ and $y'(0)=8$

$3 = c_1 + 0$; $8 = 0 + 2c_2$ so $c_1 = 3$, $c_2 = 4$

The particular solution is

$y = 3\cos 2x + 4\sin 2x$

⑩ $y'' - 10y' + 25y = 0$

a. Verify $y_1 = e^{5x}$ is a solution:

$y_1' = 5e^{5x}$, $y_1'' = 25e^{5x}$, so

$y_1'' - 10y_1' + 25y_1 = 25e^{5x} - 50e^{5x} + 25e^{5x} = 0.$

b. Verify $y_2 = xe^{5x}$ is a solution:

$y_2' = e^{5x} + 5xe^{5x}$. $y_2'' = 5e^{5x} + (5e^{5x} + 25xe^{5x})$

$= 10e^{5x} + 25xe^{5x}$

$y_2'' - 10y_2' + 25y = 10e^{5x} + 25xe^{5x} - 10(e^{5x} + 5xe^{5x}) + 25xe^{5x} = 0$

so y_2 satisfies $y'' - 10y' + 25y = 0$.

Section 3.1

(10) cont'd

(c) Let $y = c_1 e^{5x} + c_2 x e^{5x}$

then $y' = 5c_1 e^{5x} + c_2 e^{5x} + 5c_2 x e^{5x}$

Plug in $y(0) = 3$ and $y'(0) = 13$

$3 = c_1 + 0 \qquad 13 = 5c_1 + c_2 + 0$

so $c_1 = 3$ and $c_2 = -2$

The particular solution is

$$y = 3e^{5x} - 2xe^{5x}$$

(17) Let $y = \frac{1}{x}$. Then $y' = -1/x^2$.

$y' + y^2 = \frac{-1}{x^2} + \frac{1}{x^2} = 0$, so $y = 1/x$ is a solution of $y' + y^2 = 0$.

Now consider $y = \frac{c}{x}$. $y' = \frac{-c}{x^2}$.

Then $y' + y^2 = \frac{-c}{x^2} + \frac{c^2}{x^2} = \frac{c^2 - c}{x^2}$.

This is <u>not</u> $= 0$ (unless $c = 0$ or 1), so

$y = \frac{c}{x}$ is not a solution of $y' + y^2 = 0$

(Notice $y' + y^2 = 0$ is not linear, so the Superposition Theorem does not apply).

Section 3.1

(18) Consider $y = x^3$.
$y' = 3x^2$ and $y'' = 6x$.

$yy'' = x^3(6x) = 6x^4$, so $y = x^3$ is a solution of $yy'' = 6x^4$.

Now consider $y = cx^3$. $y' = 3cx^2$, $y'' = 6cx$.
Then $yy'' = (cx^3)(6cx) = 6c^2x^4$.
So $y = cx^3$ is not a solution of $yy'' = 6x^4$ (unless $c = 1$ or -1).

Since $yy'' = 6x^4$ is not linear, this does not violate the Superposition Theorem.

(19) Let $y_1 \equiv 1$. Then $y_1' \equiv 0$ and $y_1'' \equiv 0$.
So $y_1 y_1'' + (y_1')^2 = 1 \cdot 0 + 0^2 = 0$ so y_1 is a solution of $yy'' + (y')^2 = 0$

Let $y_2 = \sqrt{x}$. Then $y_2' = \frac{1}{2}x^{-1/2}$ and $y_2'' = -\frac{1}{4}x^{-3/2}$
so $y_2 y_2'' + (y_2')^2 = x^{1/2}\left(-\frac{1}{4}x^{-3/2}\right) + \left(\frac{1}{2}x^{-1/2}\right)^2 = -\frac{1}{4}x^{-1} + \frac{1}{4}x^{-1} = 0.$
Therefore $y_2 = \sqrt{x}$ is also a solution of $yy'' + (y')^2 = 0$

$\rightarrow$

Section 3.1

19 cont'd.

Now let $y_3 = 1 + \sqrt{x}$.

$y_3' = \frac{1}{2} x^{-1/2}$ and $y_3'' = \frac{-1}{4} x^{-3/2}$

Then $y_3 y_3'' + (y_3')^2 = (1 + x^{1/2})\left(\frac{-1}{4} x^{-3/2}\right) + \left(\frac{1}{2} x^{-1/2}\right)^2$

$$= -\frac{1}{4} x^{-3/2}.$$

So $y_3 = y_1 + y_2$ is not a solution of

$yy'' + (y')^2$.

Since $yy'' + (y')^2$ is not linear, this does not contradict the Superposition Theorem.

24 $f(x) = \sin^2 x$; $g(x) = 1 - \cos 2x$ are linearly dependent on the real line because $f(x) = \frac{1}{2} g(x)$, so f is a constant multiple of g.

25 $f(x) = e^x \sin x$ and $g(x) = e^x \cos x$ are linearly independent on the real line because neither is a constant multiple of the other.

Section 3.1

(27) We know y_p is a solution of $y'' + py' + qy = f(x)$, so

(1) $y_p'' + p(x)y_p' + q(x)y_p = f(x)$.

We also know y_c is a solution of $y'' + p(x)y' + q(x)y = 0$, so

(2) $y_c'' + p(x)y_c' + q(x)y_c = 0$.

Now put $y_c + y_p$ into the left hand side of the equation

$$(y_c + y_p)'' + p(x)(y_c + y_p)' + q(x)(y_c + y_p)$$

$$= (y_c'' + p(x)y_c' + q(x)y_c) + (y_p'' + p(x)y_p' + q(x)y_p)$$

Using (1) and (2), this $= 0 + f(x) = f(x)$.

Therefore $y = y_c + y_p$ is a solution of $y'' + py' + qy = f(x)$.

Section 3.1

29 Consider $y_1 = x^2$

$y_1' = 2x$ and $y_1'' = 2$, so

$$x^2 y_1'' - 4x\, y_1' + 6y_1 = x^2(2) - 4x(2x) + 6x^2 = 0$$

also $y_1(0) = 0$ and $y_1'(0) = 0$.

Consider $y_2 = x^3$. $y_2' = 3x^2$ and $y_2'' = 6x$.

so $$x^2 y_2'' - 4x\, y_2' + 6y_2 = x^2(6x) - 4x(3x^2) + 6x^3 = 0$$

Also $y_2(0) = 0$ and $y_2'(0) = 0$.

Both y_1 and y_2 are solutions of the initial value problem $x^2 y'' - 4xy' + 6y = 0$, $y(0) = y'(0) = 0$

This does not contradict theorem 2 because for theorem 2, the D.E. must be in the form $y'' + p(x)y' + q(x)y = \cancel{0}\; f(x)$ with p, q, f continuous on an open interval containing $a = 0$.

If we put our equation in this form by dividing by x^2, $\left(y'' - \frac{4}{x}y' + \frac{6}{x^2} = 0\right)$, we see that $p(x)$ and $q(x)$ are not continuous at 0.