# Iode Project III

1.a. $x'' + \frac{3}{2}x' + \frac{1}{2}x = 0$ ; $2x'' + 3x' + x = 0$

Characteristic equation $2r^2 + 3r + 1 = 0$

$(2r+1)(r+1) = 0$. $r = -\frac{1}{2}$, $r = -1$

General solution $x(t) = c_1 e^{-\frac{1}{2}t} + c_2 e^{-t}$

1c. It appears that all of the solutions decay. This is in fact the case, because for any choice of constants $c_1$ and $c_2$,

$$\lim_{t\to\infty} \left(c_1 e^{-\frac{1}{2}t} + c_2 e^{-t}\right) = 0.$$

2a. $x'' - (\sqrt{3})x' - .25x = 0$ ; $r^2 - \sqrt{3}r - .25 = 0$

$$r = \frac{\sqrt{3} \pm \sqrt{3 + 4(.25)}}{2} = \cancel{\frac{\sqrt{3} \pm \sqrt{2}}{2}} \quad \frac{\sqrt{3} \pm 2}{2} = \frac{\sqrt{3}}{2} \pm 1$$

General solution $x = c_1 e^{(\sqrt{3}/2 + 1)t} + c_2 e^{(\sqrt{3}/2 - 1)t}$

2b. All of my examples appeared to grow.

2c. Observe that $\sqrt{3}/2 + 1 > 0$ and $\frac{\sqrt{3}}{2} - 1 < 0$.

$$\lim_{t\to\infty} \left(c_1 e^{(\sqrt{3}/2 + 1)t} + c_2 e^{(\sqrt{3}/2 - 1)t}\right) = \begin{cases} +\infty & \text{if } c_1 > 0 \\ -\infty & \text{if } c_1 < 0 \\ 0 & \text{if } c_1 = 0 \end{cases}$$

So the solutions all grow, except when $c_1 = 0$. Notice this is still an infinite number of cases; $c_1 = 0$, $c_2 =$ anything.

3a. $x'' - .5x' + .5x = 0$; $\quad r^2 - .5r + .5 = 0$

$2r^2 - r + 1 = 0$; $\quad r = \dfrac{1 \pm \sqrt{1-8}}{4} = \dfrac{1}{4} \pm \dfrac{\sqrt{7}}{4} i$

General solution: $x = e^{\frac{1}{4}t}\left(c_1 \cos \frac{\sqrt{7}}{4}t + c_2 \sin \frac{\sqrt{7}}{4}t\right)$

3b. All of my examples appeared to oscillate while growing.

3c. Since the solution is $e^{\frac{1}{4}t}$ times a combination of sin and cos functions, all the solutions will grow while oscillating except for <u>one</u> solution, namely when $c_1 = c_2 = 0$, so that $x \equiv 0$.

4.a. $x^{(4)} + 16x'' + 100x = 0$, $\quad r^4 + 16r^2 + 100 = 0$

Using the hint, $(r^2 + 2r + 10)(r^2 - 2r + 10) = 0$

Using the quadratic formula on $r^2 + 2r + 10 = 0$ and on $r^2 - 2r + 10 = 0$,

$r = \dfrac{-2 \pm \sqrt{4-40}}{2} = \dfrac{-2 \pm \sqrt{-36}}{2} = \dfrac{-2 \pm 6i}{2} = \boxed{-1 \pm 3i}$

and

$r = \dfrac{2 \pm \sqrt{4-40}}{2} = \boxed{1 \pm 3i}$ General solution:

$x(t) = e^{-t}(c_1 \cos 3t + c_2 \sin 3t) + e^{t}(c_3 \cos 3t + c_4 \sin 3t)$

4b. The first part, $e^{-t}(c_1 \cos 3t + c_2 \sin 3t)$ will decay while oscillating.

The second part, $e^{t}(c_3 \cos 3t + c_4 \sin 3t)$, will grow while oscillating (unless $c_3 = c_4 = 0$).

So their sum will grow while oscillating 100% of the time, but there are still an infinite number of cases in which it decays while oscillating ($c_3 = c_4 = 0$, $c_1, c_2 =$ any real numbers).

In <u>one</u> solution ($c_1 = c_2 = c_3 = c_4$), it doesn't oscillate at all. (The trivial solution).

5a. $r_1 < r_2 < 0$ decays

b. $r_1 < 0 < r_2$ grows

c. $0 < r_1 < r_2$ grows

d. $r_1, r_2 = \alpha \pm \beta i$, $\alpha < 0$ decays while oscillating

e. $r_1, r_2 = \alpha \pm \beta i$, $\alpha = 0$ oscillates

f. $r_1, r_2 = \alpha \pm \beta i$, $\alpha > 0$ grows while oscillating

# HW #6 Solutions

## 3.2 #1

$\frac{-5}{2} f(x) + \frac{8}{3} g(x) + 1 \quad h(x)$

$= -5x + 8x^2 + (5x - 8x^2) = 0$, so $f, g, h$ are ~~not~~ linearly dependent.

## 3.2 #4

$-\frac{1}{17} f(x) + \frac{1}{2} g(x) + \frac{1}{3} h(x)$

$= -1 + \sin^2 x + \cos^2 x = -1 + 1 = 0$, so

$f, g, h$ are linearly dependent.

## 3.2 #21

General solution is $y = y_p + y_c = 3x + c_1 \cos x + c_2 \sin x$

$y' = 3 - c_1 \sin x + c_2 \cos x$

$2 = y(0) = 0 + c_1 + 0$ so $c_1 = 2$

$-2 = y'(0) = 3 - 0 + c_2$ so $c_2 = -5$

$$\boxed{y = 3x + 2\cos x + (-5)\sin x}$$

3.2 # 27

If $f_1, f_2, f_3$ are linearly dependent, then there are constants $c_1, c_2, c_3$, not all $=0$, such that $c_1 f_1 + c_2 f_2 + c_3 f_3 \equiv 0$ for all $x$

In this case $c_1 + c_2 x + c_3 x^2 = 0$ for all $x$
Since the left side is a polynomial, it can be identically 0 only if all the coefficients are 0.

$\boxed{\text{OR}}$ take the approach suggested in the book.

(1) $c_1 + c_2 x + c_3 x^2 = 0$.
(2) $c_2 + 2c_3 x = 0$
(3) $2c_3 = 0$.

Take derivative of both sides twice

Putting $x=0$ into (1), (2), (3) gives $c_1 = c_2 = c_3 = 0$.

So the only way for $c_1 f_1 + c_2 f_2 + c_3 f_3 = 0$ to hold is $c_1 = c_2 = c_3 = 0$. This means $f_1, f_2, f_3$ are linearly independent.

3.2 #29 for $n=2$

Suppose $c_0 f_0 + c_1 f_1 + c_2 f_2 = 0$ for all $x$.

then $c_0 e^{rx} + c_1 x e^{rx} + c_2 x^2 e^{rx} = 0$

so $e^{rx}(c_0 + c_1 x + c_2 x^2) = 0$ for all $x$.

Since $e^{rx} \neq 0$, it must be that

$$c_0 + c_1 x + c_2 x^2 = 0. \quad \text{for all } x$$

From #27, $1, x, x^2$ are linearly independent, so $c_0 + c_1 x + c_2 x^2$ occurs only if $c_0 = c_1 = c_2 = 0$.

Therefore $c_0 f_0 + c_1 f_1 + c_2 f_2 = 0$ for all $x$ occurs only if $c_0 = c_1 = c_2 = 0$.

Therefore ~~$c_0, c_1$.~~ $f_0, f_1, f_2$ are linearly independent.

3.3 #3 $y'' + 3y' - 10y = 0$ $\quad r^2 + 3r - 10 = 0$

$(r+5)(r-2) = 0$ $\quad r = -5, 2$ $\quad \boxed{y = c_1 e^{-5x} + c_2 e^{2x}}$

3.3 #9

$y'' + 8y' + 25y = 0$ $\quad r^2 + 8r + 25 = 0$

$r = \dfrac{-8 \pm \sqrt{64-100}}{2} = \dfrac{-8 \pm \sqrt{-36}}{2} = \dfrac{-8 \pm 6i}{2}$

$= -4 \pm 3i$ $\quad \boxed{y = e^{-4x}(c_1 \cos 3x + c_2 \sin 3x)}$

3.3 #21

$y'' - 4y' + 3y = 0$ $\quad r^2 - 4r + 3 = 0$ $\quad (r-3)(r-1) = 0$

$r = 3, 1$ $\quad y = c_1 e^{3x} + c_2 e^{x}$ ; $\quad y' = 3c_1 e^{3x} + c_2 e^{x}$

$\left.\begin{array}{l} 7 = y(0) = c_1 + c_2 \\ 11 = y'(0) = 3c_1 + c_2 \end{array}\right\}$ subtract. $\quad 4 = 2c_1$ $\quad c_1 = 2$

then $c_2 = 5$

$\cancel{y = 2e^{3x} +}$ $\quad \boxed{y = 2e^{3x} + 5e^{x}}$

## 3.3 # 25

$3y^{(3)} + 2y'' = 0 \qquad 3r^3 + 2r^2 = 0$

$r^2(3r+2) = 0, \qquad r = 0, 0, -2/3$

$y = c_1 1 + c_2 x + c_3 e^{-2/3 x} = \boxed{c_1 + c_2 x + c_3 e^{-2/3 x}}$

Note: $1 = e^{0x} \quad x = xe^{0x}$ from $r=0$ roots

## 3.3 # 39

$y(x) = (A + Bx + Cx^2)e^{2x}$

$= Ae^{2x} + Bxe^{2x} + Cx^2e^{2x}$

The roots are $r = 2, 2, 2$

Characteristic equation $(r-2)(r-2)(r-2) = 0$

$r^3 + 3r^2(-2) + 3r(-2)^2 + (-2)^3 = 0$

$r^3 - 6r^2 + 12r - 8 = 0$

Note: I have used $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$$\boxed{y''' - 6y'' + 12y' - 8y = 0}$$