~~242~~ 385 Assignment #8 Solutions

3.6/2 $\quad x'' + 4x = 5\sin 3t \qquad x(0) = x'(0) = 0$

Solve: $x'' + 4x = 0 \qquad r^2 + 4 = 0 \qquad r = \pm 2i$

$$x_c = c_1 \cos 2t + c_2 \sin 2t$$

Undetermined coefficients: $x_p = A\sin 3t$

(Can leave off $\cos 3t$ since no x' term)

$x_p'' = -9A\sin 3t$

$x_p'' + x_p = -9A\sin 3t + 4A\sin 3t = -5A\sin 3t$.

This should equal $5\sin 3t$, so $A = -1$

$x_p = -\sin 3t$

$x = -\sin 3t + c_1\cos 2t + c_2 \sin 2t$

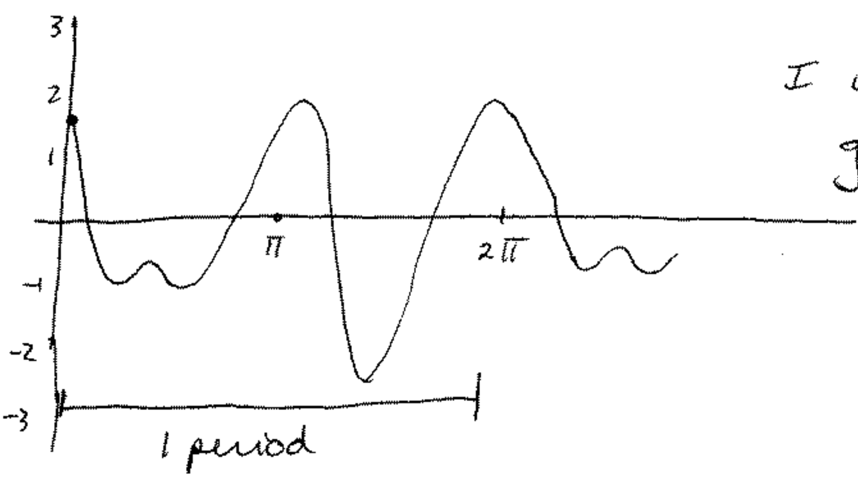
$0 = x(0) = c_1$

$x' = -3\cos 3t + 2c_2\cos 2t$

$0 = x'(0) = -3 + 2c_2 \Rightarrow c_2 = 3/2$

$$\boxed{x = -\sin 3t + \tfrac{3}{2}\cos 2t}$$

Period is least common multiple of $\frac{2\pi}{3}$ and π : 2π



I used a graphing calculator

3.6/7 $\quad x'' + 4x' + 4x = 10\cos 3t$

Solve $x'' + 4x' + 4x = 0$

$r^2 + 4r + 4 = 0 \qquad (r+2)^2 = 0 \qquad r = -2, -2$

$$\boxed{X_c = c_1 e^{-2t} + c_2 t e^{-2t}}$$

Undetermined coefficients:

$$X_{sp} = A\cos 3t + B\sin 3t$$

$$X_{sp}' = -3A\sin 3t + 3B\cos 3t$$

$$X_{sp}'' = -9X_{sp}$$

$$X_{sp}'' + 4X_{sp}' + 4X_{sp} = -5X_{sp} + 4X_{sp}'$$

$$= -5A\cos 3t - 5B\sin 3t - 12A\sin 3t + 12B\cos 3t$$

$$= (12B - 5A)\cos 3t + (-5B - 12A)\sin 3t.$$

This should equal $10\cos 3t$, so

$5\times(12B - 5A = 10)$

$12\times(-5B - 12A = 0)$

$$-169A = 50 \qquad A = \frac{-50}{169}.$$

$$B = -\frac{12}{5}A = \frac{120}{169}$$

$$C = \sqrt{A^2+B^2} = \frac{130}{169}$$

$$\boxed{X_{sp}(t) = \frac{130}{169}\cos(3t - \alpha)}$$

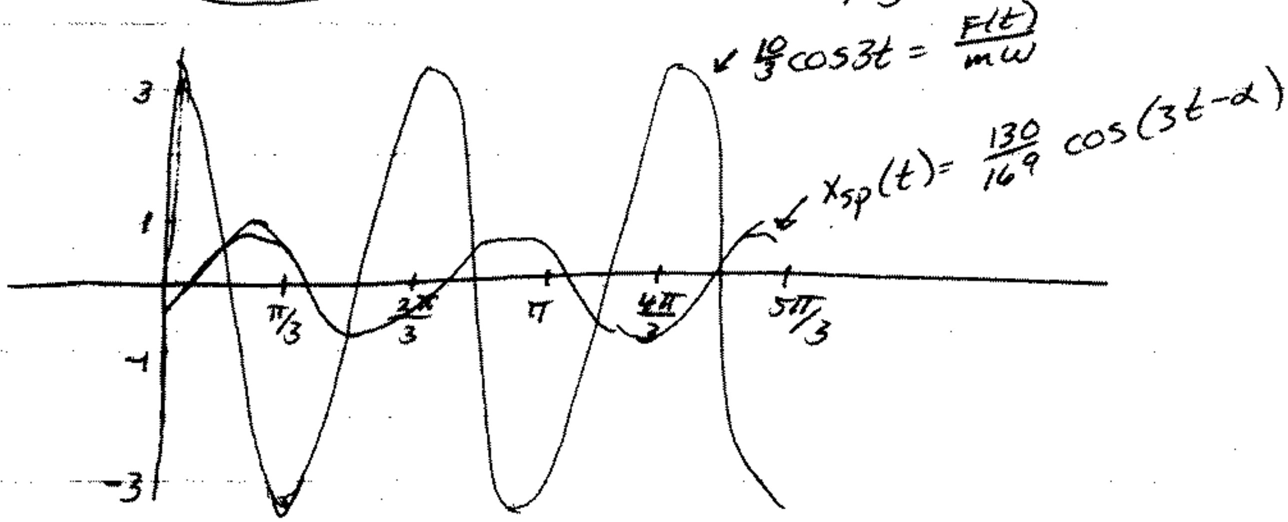
where $\cos\alpha = -\frac{5}{13}$

$\sin\alpha = 12/13$

$\alpha \approx 1.97$ radians

cont'd →

3.6/7 cont'd $\quad \dfrac{F(t)}{m\omega} = \dfrac{10\cos 3t}{1\cdot 3} = \dfrac{10}{3}\cos 3t$



3.6/11 $\quad x'' + 4x' + 5x = 10\cos 3t \qquad x(0) = x'(0) = 0$

$r^2 + 4r + 5 = 0$

$r = \dfrac{-4 \pm \sqrt{16-20}}{2} = \dfrac{-4 \pm 2i}{2} = -2 \pm i$

$$\boxed{x_c = e^{-2t}(c_1\cos t + c_2 \sin t)}$$

$x_p = A\cos 3t + B\sin 3t$

$x_p' = -3A\sin 3t + 3B\cos 3t$

$x_p'' = -9x_p$

$x_p'' + 4x_p' + 5x_p = -4x_p + 4x_p'$

$= -4A\cos 3t - 4B\sin 3t - 12A\sin 3t + 12B\cos 3t$

This should equal $10\cos 3t$, so

$-4A + 12B = 10$

$3\times(-12A - 4B = 0) \Rightarrow 4B = -12A = 3 \qquad B = 3/4$

$-40A = 10$

$A = -1/4$

$$\boxed{x_p = -\tfrac{1}{4}\cos 3t + \tfrac{3}{4}\sin 3t}$$

3.6/11 cont'd

$$x = x_p + x_c = e^{-2t}(c_1 \cos t + c_2 \sin t) - \tfrac{1}{4}\cos 3t + \tfrac{3}{4}\sin 3t$$

Put in $x(0)=0$:

$$0 = c_1 - \tfrac{1}{4} \quad \text{so} \quad c_1 = 1/4$$

$$x' = -2e^{-2t}\left(\tfrac{1}{4}\cos t + c_2 \sin t\right) + e^{-2t}\left(-\tfrac{1}{4}\sin t + c_2 \cos t\right) + 3/4 \sin 3t + 9/4 \cos 3t$$

Put in $x'(0)=0$:

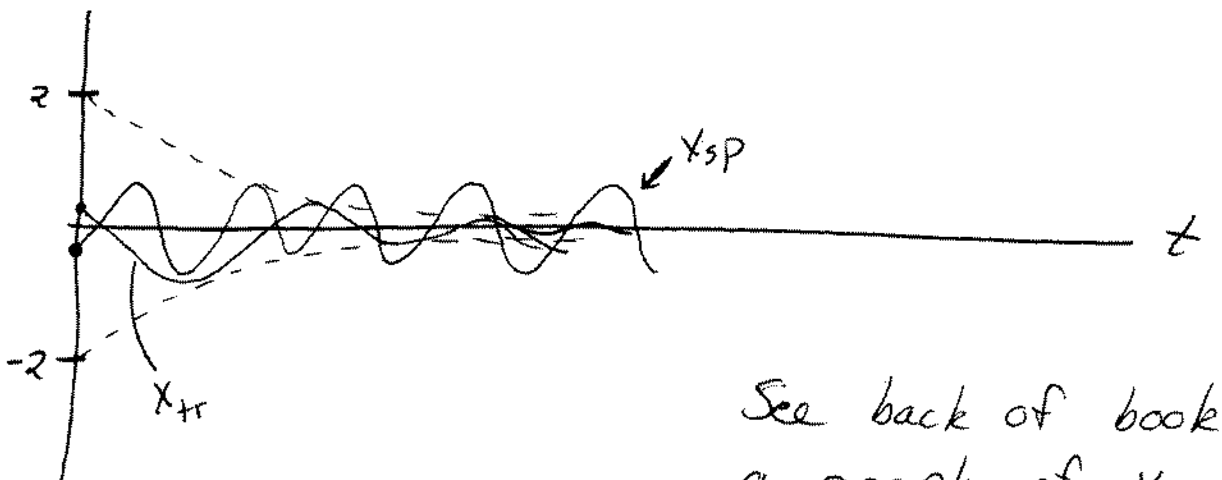
$$0 = -2(1/4) + c_2 + 9/4 \quad \text{so} \quad c_2 = -7/4$$

$$x_{tr} = x_c = e^{-2t}\left(\tfrac{1}{4}\cos t - \tfrac{7}{4}\sin t\right) = \tfrac{5\sqrt{2}}{4} e^{-2t}\cos(t-\beta)$$

$\beta \approx -1.43$

$$x_{sp} = x_p = -\tfrac{1}{4}\cos 3t + 3/4 \sin 3t = \tfrac{\sqrt{10}}{4}\cos(3t-\alpha)$$

where $\cos\alpha = -1/\sqrt{10}$ $\quad \sin\alpha = 3/\sqrt{10}$ $\quad \alpha \approx 1.89$



See back of book for a graph of x_{sp} and $x = x_{sp} + x_{tr}$

(3.6/15) $x'' + 2x' + 2x = 2\cos\omega t$

Undetermined coefficients:

$x_p = A\cos\omega t + B\sin\omega t$

$x_p' = -A\omega\sin\omega t + B\omega\cos\omega t$

$x_p'' = -\omega^2 x_p$

$$x_p'' + 2x_p' + 2x_p = (2-\omega^2)x_p + 2x_p'$$

$$= (2-\omega^2)A\cos\omega t + (2-\omega^2)B\sin\omega t$$
$$+ 2B\omega\cos\omega t - 2A\omega\sin\omega t.$$

Then $(2-\omega^2)A + 2\omega B = 2$

$$-2\omega A + (2-\omega^2)B = 0$$

$A = \frac{(2-\omega^2)}{2\omega}B$ so

$$\frac{(2-\omega^2)^2}{2\omega}B + 2\omega B = 2$$

$$\frac{(2-\omega^2)^2 + 4\omega^2}{2\omega}B = 2$$

$$B = \frac{4\omega}{(2-\omega^2)^2 + 4\omega^2}, \quad A = \frac{2(2-\omega^2)}{(2-\omega^2)^2 + 4\omega^2}$$

$$C(\omega) = \sqrt{A^2+B^2} = \left[(2-\omega^2)^2 + 4\omega^2\right]^{-1} \cdot 2\left[(2\omega)^2 + (2-\omega^2)^2\right]^{1/2}$$
$$= 2\left[(2-\omega^2)^2 + 4\omega^2\right]^{-1/2}$$

$$C'(\omega) = -\left[(2-\omega^2)^2 + 4\omega^2\right]^{-3/2}\underbrace{\left[2(2-\omega^2)(-2\omega) + 8\omega\right]}$$

$$-8\omega + 4\omega^3 + 8\omega = +4\omega^3$$

The only critical point of $C(\omega)$ is $\omega = 0$.

No practical resonance frequency.

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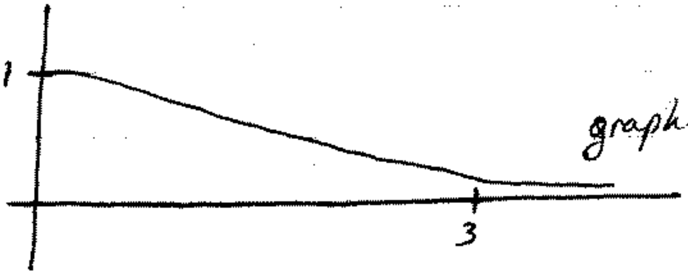
3.6/15 cont'd



graphof $C(\omega)$

3.6/18 $\quad x'' + 10x' + 650x = 100\cos\omega t$

$$x_p = A\cos\omega t + B\sin\omega t$$

$$x_p'' + 10x_p' + 650x_p$$
$$= (650-\omega^2)A\cos\omega t + (650-\omega^2)B\sin\omega t + 10\omega B\cos\omega t - 10\omega A\sin\omega t$$

Then
$$(650-\omega^2)A + 10\omega B = 100$$
$$-10\omega A + (650-\omega^2)B = 0$$

We get
$$A = \frac{(650-\omega^2)\,100}{(650-\omega^2)^2 + (10\omega)^2}$$

see p.216

$$B = \frac{10\omega \cdot 100}{(650-\omega^2)^2 + (10\omega)^2}$$

$$C = \sqrt{A^2 + B^2} = 100\left[(650-\omega^2)^2 + (10\omega)^2\right]^{-1/2}$$

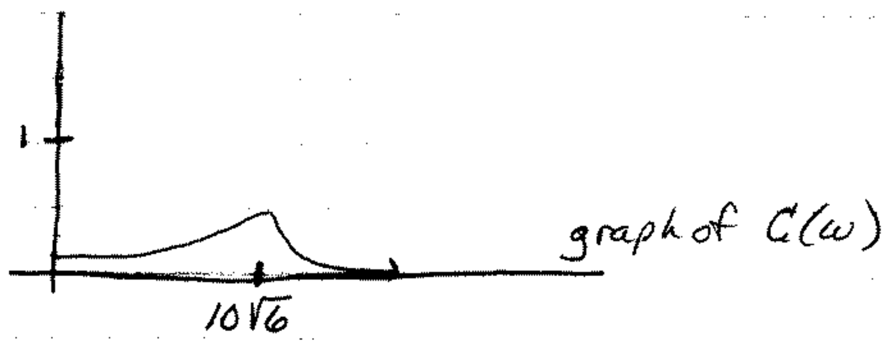
$$C'(\omega) = -50[\quad]^{-3/2}\underbrace{\left[2(650-\omega^2)(-2\omega) + 200\omega\right]}$$

$$-2600\omega + 4\omega^3 + 200\omega = 4\omega^3 - 2400\omega$$

$C'(\omega) = 0$ for $\omega^2 = 600$ $\quad \omega = 10\sqrt{6}$ resonance

Practical resonance occurs for $\omega = 10\sqrt{6}$

3.6/18 cont'd



Problem "A" $\quad 4x'' + 16x = 5\cos\omega t \qquad$ (use ω instead of a)

The solution to $4x''+16x=0$
is $c_1 \cos 2t + c_2 \sin 2t$, so the natural frequency of the system is 2 and pure resonance will occur for $\boxed{\omega = 2}$

x_p will be of the form $\{ x_p = At\cos 2t + Bt\sin 2t$

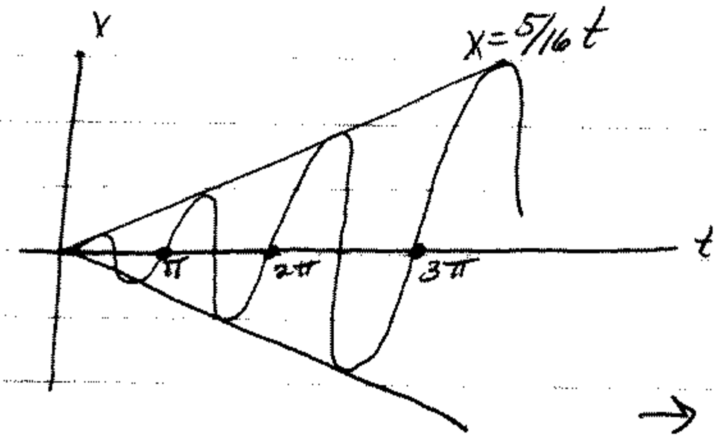
$x_p' = A\cos 2t - 2At\sin 2t + B\sin 2t + 2Bt\cos 2t$

$x_p'' = -2A\sin 2t - 2A\sin 2t - 4At\cos 2t$
$\quad + 2B\cos 2t + 2B\cos 2t - 4Bt\cancel{\cos}\sin 2t$
$\quad = -4A\sin 2t + 4B\cos 2t - 4x_p$

$4x_p'' + 16x_p = -16A\sin 2t + 16B\cos 2t \qquad$ (after cancelling)
$\qquad = 5\cos 2t$

$A = 0 \qquad B = 5/16$

$x_p = \frac{5}{16} t \sin 2t$



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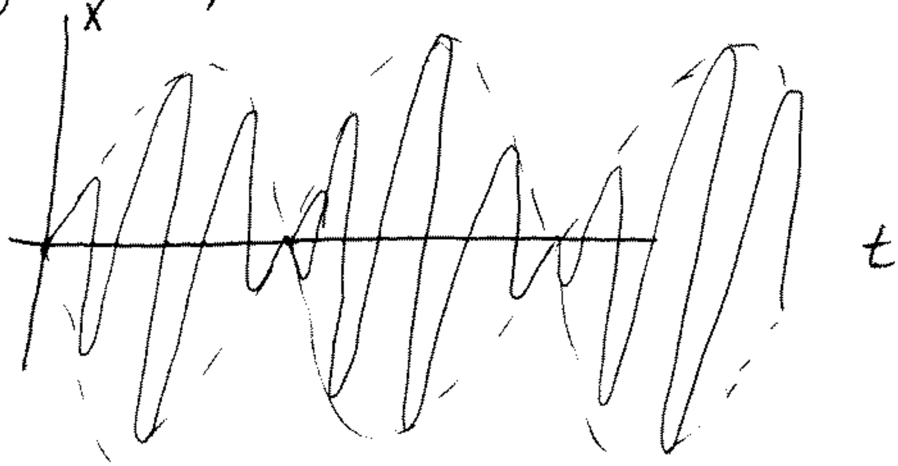
Problem "A" cont'd

If ω is close to, but not equal to, 2, there there will be "beats" with a high amplitude, like this:



3.8/1 Let $\lambda = 0$. $y'' = 0$ so $y(x) = Ax + B$

Then $y'(0) = 0$ and $y(1) = 0 \Rightarrow A = B = 0$.

So $y(x) = 0$ is the only solution.

$\lambda = 0$ is not an eigenvalue.

To find eigenvalues: assume $\lambda > 0$

$r^2 + \lambda = 0 \quad r = \pm\sqrt{+\lambda}\, i$

$y = A\cos(\sqrt{+\lambda}\, x) + B\sin(\sqrt{+\lambda}\, x)$

$y' = -A\sqrt{+\lambda}\sin(\sqrt{+\lambda}\, x) + B\sqrt{+\lambda}\cos(\sqrt{+\lambda}\, x)$

Plug in: $0 = y'(0) = B\sqrt{+\lambda}$ so $B = 0$

$0 = y(1) = A\cos(\sqrt{\lambda})$

Either $A = 0$ (trivial solution) or

$\cos(\sqrt{\lambda}) = 0 \qquad \sqrt{\lambda} = \frac{n\pi}{2}$, n odd

$\lambda = \frac{n^2\pi^2}{4}$, n odd $\rightarrow$

3.8/11 cont'd

The eigenvalues are $\lambda = \frac{n^2\pi^2}{4}$, n odd

The associated eigenfunctions are

$$y(x) = A\cos\left(\frac{n\pi}{2}x\right) \quad (n \text{ odd})$$

3.8/13 (a) Let $\lambda = 1$. $y'' + 2y' + y = 0$

$r^2 + 2r + 1 = 0 \quad (r+1)^2 = 0 \quad r = -1, -1$

$y(x) = Ae^{-x} + Bxe^{-x}$

Plug in: $0 = y(0) = A$ so $A = 0$

$0 = y(1) = B \cdot 1 \cdot e^{-1}$ so $B = 0$

The only solution is the trivial one, so $\lambda = 1$ is not an eigenvalue.

(b) Let $\lambda < 1$. $y'' + 2y' + \lambda y = 0$

$r^2 + 2r + \lambda = 0$

$r = \frac{-2 \pm \sqrt{4 - 4\lambda}}{2}$ Since $\lambda < 1$, $-\lambda > 1$, $-4\lambda > -4$

and $4 - 4\lambda > 0$

so the roots are real and distinct.

$y(x) = Ae^{r_1 x} + Be^{r_2 x}$ where r_1, r_2 are the roots.

Plug in: $0 = y(0) = A + B$

$0 = y(1) = Ae^{r_1} + Be^{r_2}$

Since $r_1 \neq r_2$, when we solve we get $A = B = 0$.

The only solution is the trivial one, so $\lambda < 1$ is not an eigenvalue. →

3.8/13 cont'd

(c) Let $\lambda > 1$. $y'' + 2y' + \lambda y = 0$

$r^2 + 2r + \lambda = 0$

$r = -1 \pm \sqrt{1-\lambda} = -1 \pm (\sqrt{\lambda - 1})i$ since $\lambda > 1$.

complex roots.

$y = e^{-x}(A\cos(\sqrt{\lambda-1}\,x) + B\sin(\sqrt{\lambda-1}\,x))$

Plug in: $0 = y(0) = 1\cdot(A + 0)$ so $A = 0$

$0 = y(1) = e^{-1}\cdot B\sin(\sqrt{\lambda-1})$.

λ is an eigenvalue if $\sin(\sqrt{\lambda-1}) = 0$

$\sqrt{\lambda-1} = n\pi$ (n a positive integer)

$\lambda - 1 = n^2\pi^2$ $\lambda = n^2\pi^2 + 1$

The eigenvalues are $\lambda = n^2\pi^2 + 1$ for n positive integer.

The associated eigenfunctions are

$y = Be^{-x}\sin(\sqrt{\lambda-1}\,x)$, which equals

$Be^{-x}\sin(n\pi x)$

3.8/16 (a) The equation (p. 236) is

$$EI\,y^{(4)} = w$$

With initial conditions (p. 237)

$$y(0)=0 \quad y'(0)=0 \quad y(L)=0 \quad y'(L)=0.$$

Solve the equation by succesive integrations:

$$y^{(4)} = \frac{w}{EI}$$

$$y^{(3)} = \frac{w}{EI}x + c_1$$

$$y'' = \frac{w}{2EI}x^2 + c_1 x + c_2$$

$$y' = \frac{w}{6EI}x^3 + \frac{c_1}{2}x^2 + c_2 x + c_3$$

$$y = \frac{w}{24EI}x^4 + \frac{c_1}{6}x^3 + \frac{c_2}{2}x^2 + c_3 x + c_4$$

Plugging in the initial conditions, we end up with

$$y(x) = \frac{w}{24EI}\left(x^4 - 2Lx^3 + L^2x^2\right)$$

(b) $$y'(x) = \frac{w}{24EI}\left(4x^3 - 6Lx^2 + 2L^2x\right)$$

$$= \frac{w}{24EI}\cdot 2x\left(2x^2 - 3Lx + L^2\right)$$

3.8/16 cont'd

$$y'(x) = \frac{w}{24EI} \cdot 2x\,(2x-L)(x-L)$$

so $y'(x) = 0$ for $x=0$, $x = L/2$, $x = L$.

at $x=0$ and $x=L$, $y(x)=0$ (initial conditions!)
The maximum deflection occurs
right in the middle, at $x=L/2$ and is

$$y(L/2) = \frac{w}{24EI}\left(\left(\frac{L}{2}\right)^4 - 2L\left(\frac{L}{2}\right)^3 + L^2\left(\frac{L}{2}\right)^2\right)$$

$$= \frac{w}{24EI} L^4 \left(\frac{1}{16} - \frac{1}{4} + \frac{1}{4}\right)$$

$$= \frac{wL^4}{384EI}.$$