Lecture Notes

Math 403
april 16 – 23

We start at Section 4.5. Prior to now, we have seen that an isometry falls into one of four categories:

(1) no fixed point
(2) exactly one fixed point
(3) two fixed points (but not = identity) ⇒ there is a whole line of fixed points and the isometry must be REFLECTION over the line.
(4) three noncollinear fixed points ⇒ every point is fixed and the isometry is the IDENTITY.

It remains to somehow list all isometries in category (1) and (2), zero or one fixed point.

$\boxed{\text{Section 4.5}}$

Theorem 4.18 If α is an isometry with exactly one fixed point P, then $\alpha = \sigma_n \sigma_m$, where m and n are two lines intersecting at P.

Proof: We will discuss the proof in the book.

True or False? If $\alpha = \sigma_n \sigma_m$ and $\alpha = \sigma_{n'} \sigma_{m'}$, then n must $= n'$ and m must $= m'$.

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The following theorem will be crucial in our classification of isometries:

<u>Theorem 4.20</u> Every isometry can be written as the composition of 3 or fewer ~~isometries~~ reflections.

<u>Proof</u> The identity is the composition of two reflections, namely any reflection composed with itself.

Let α be any isometry which is not the identity. Then there is some point P which is not fixed by α. Denote $Q = \alpha(P)$ and note $P \neq Q$. Let m be the perpendicular bisector of P and Q. Then, by definition of the reflection σ_m,

$$\sigma_m(Q) = P.$$

Since $Q = \alpha(P)$, this says

$$\sigma_m \circ \alpha(P) = P,$$

so P is a fixed point of $\sigma_m \circ \alpha$.

If $\sigma_m \circ \alpha$ has other fixed points besides P, then it must be (see p.1 of these notes)

<u>either</u> a reflection σ_n, in which case

$$\sigma_m \circ \alpha = \sigma_n, \quad \text{so} \quad \underbrace{\sigma_m \circ \sigma_m}_{\iota} \circ \alpha = \sigma_m \circ \sigma_n \quad \text{so} \quad \alpha = \sigma_m \circ \sigma_n,$$

composition of <u>two</u> reflections.

<u>or</u> the identity, in which case

$$\sigma_m \circ \alpha = \iota, \quad \text{so} \quad \underbrace{\sigma_m \circ \sigma_m}_{\iota} \circ \alpha = \sigma_m \circ \iota$$

so $\alpha = \sigma_m$, a reflection.

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If $\sigma_m \circ \alpha$ has only P as fixed point, then by Theorem 4.18 there are lines l and n such that

$$\sigma_m \circ \alpha = \sigma_l \circ \sigma_n \,, \quad \text{so}$$

$$\underbrace{\sigma_m \circ \sigma_m}_{\iota} \circ \alpha = \sigma_m \circ \sigma_l \circ \sigma_n$$

so $\alpha = \sigma_m \circ \sigma_l \circ \sigma_n$.

Thus any isometry can be written as a composition of one, two or three isometries. ∇.

$\boxed{\text{Section 4.6}}$ We will omit most of this section for lack of time. However Exercise 4.8 is assigned. No proof is required, but you should describe what the isometry $\sigma_p \sigma_l$ does in as much detail as possible. E.g., if it is a rotation, then through what ~~center~~ angle and with what center? If it is a translation, what is the translation vector? If it is a reflection, what is the line of reflection? Etc.

$\boxed{\text{Section 4.7}}$ Translations

The main point of this section is to understand how translations (an example of an isometry with no fixed point) are related to reflections. From Theorem 4.20, we now know that a translation can be written as the composition of 3 or fewer reflections. The following theorems give more detail.

<u>Theorem 4.24</u> Let m and n be parallel lines. Then $\sigma_n \sigma_m = \tau_A$ where A is perpendicular to m and n and has length twice the distance from m to n.

<u>Theorem 4.25</u> For any translation τ_A, we can find parallel lines m and n such that $\tau_A = \sigma_n \circ \sigma_m$. m and n will be $\perp$ to A and the distance between them will be half the length of A.

<u>Proofs</u>: The proofs are not so very hard, but I will skip them and instead we will convince ourselves by means of pictures that these two theorems are true. - to be done in class.

Notice that if we are given vector A and <u>any</u> line p, then there is a unique line q which is parallel to p such that $\tau_A = \sigma_q \sigma_p$. This fact is used in the proof of the next theorem.

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Theorem 4.26 [This will be used in the proof of the classification theorems 4.36, 4.37]

Let m, n, p be parallel lines. Then there is a line q, parallel to m, n, p, such that
$$\sigma_n \circ \sigma_m \circ \sigma_p = \sigma_q.$$

Proof: $\sigma_m \circ \sigma_n =$ a translation τ_A, where $A \perp$ to m and n. Since $A \perp p$ also, there is some q, parallel to p, with $\tau_A = \sigma_q \circ \sigma_p$.

Since $\sigma_m \circ \sigma_n = \tau_A$ and $\sigma_q \circ \sigma_p = \tau_A$,

$$\sigma_m \circ \sigma_n = \sigma_q \circ \sigma_p$$
$$\sigma_m \circ \sigma_n \circ \sigma_p = \sigma_q \circ \underbrace{\sigma_p \circ \sigma_p}_{\iota}$$
$$\sigma_m \circ \sigma_n \circ \sigma_p = \sigma_q \qquad \square.$$