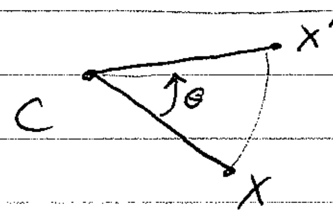
Math 403 Lecture Notes

Section 4.8 Rotations.

<u>Definition</u> The definition of a rotation agrees with our intuitive understanding. Given a center C and an angle θ (could be negative), $\rho_{C,\theta}$ is rotation with center C by θ degrees counterclockwise.
$\rho_{C,\theta}(C) = C$ and for $X \neq C$,

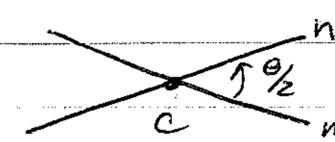
$X' = \rho_{C,\theta}(X)$ is a point such that
$$d(C,X) = d(C,X') \text{ and}$$
$$\measuredangle(X-C, X'-C) = \theta$$



<u>Theorem 4.27</u> A rotation is an isometry.

<u>Proof</u> We'll accept this theorem without proof as it is intuitively clear that rotating preserves all distances.

<u>Theorem 4.28</u> Given a rotation $\rho_{C,\theta}$, let m and n be any two lines through C so that $\measuredangle(m,n) = \frac{\theta}{2}$



Then $\rho_{C,\theta} = \sigma_n \sigma_m$

Proof of Theorem 4.28

We are given C, lines m and n intersecting at C with $\measuredangle(m,n) = \theta/2$.

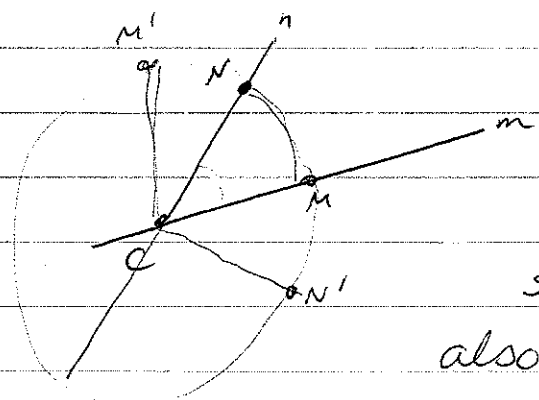
Let M be on m, N on ~~n~~ n, so that $d(C,M) = d(C,N)$.

Let $N' = \sigma_m(N)$ and $M' = \sigma_n(M)$.

We will show that $\rho_{C,\theta}$ and $\sigma_n\sigma_m$ agree on the three non-collinear points C, M, N', which by Exercise 4.2 will imply $\rho_{C,\theta} = \sigma_n\sigma_m$.

C is fixed by both $\rho_{C,\theta}$ and $\sigma_n\sigma_m$.

$d(C,M) = d(C,M')$ and ~~$\angle($~~, since isometries preserve angles,



$$\angle(M-C, M'-C) =$$
$$\angle(M-C, N-C) + \angle(N-C, M'-C)$$
$$= \theta/2 + \theta/2 = \theta$$

so $\rho_{C,\theta}(M) = M'$.

also $\sigma_n\sigma_m(M) = \sigma_n(M) = M'$.

By argument similar to above, $\rho_{C,\theta}(N') = N$ and $\sigma_n\sigma_m(N') = \sigma_n(N) = N$

$\therefore\ \rho_{C,\theta} = \sigma_n\sigma_m$ $\square$

Thm 4.30 An isometry with exactly one fixed point C is a rotation with center C.

Proof: Theorem 4.28 and Theorem 4.18 $\square$.

Theorem 4.29 (This will be needed in the classification theorem for isometries).

~~Proof~~ Let m, n, p be lines which all intersect at a point ~~Z~~ C. ~~Let~~ Then there is a line q, through C, so that

$$\sigma_n \circ \sigma_m \circ \sigma_p = \sigma_q.$$

Proof $\sigma_n \circ \sigma_m$ is a rotation $\rho_{C,\theta}$, where $\theta/2 = \angle(m,n)$. Let q be a line through C such that $\angle(p,q) = \theta/2$

Then, by Theorem 4.28, $\rho_{C,\theta} = \sigma_q \circ \sigma_p$.

Since $\rho_{C,\theta}$ equals both $\sigma_n \circ \sigma_m$ and $\sigma_q \sigma_p$,

$\sigma_n \circ \sigma_m = \sigma_q \circ \sigma_p$, so

$$\sigma_n \circ \sigma_m \circ \sigma_p = \sigma_q \circ \sigma_p \circ \sigma_p = \sigma_q \circ \iota = \sigma_q \quad \square.$$