Math 403 Lecture Notes.

<u>Exercise</u>: Let m be the x-axis, n the y-axis, and p the line $y=x$. Find a line q such that
$$\sigma_n \circ \sigma_m \circ \sigma_p = \sigma_q \qquad \text{(See Theorem 4.29)}$$

Section 4.9 Glide Reflections

<u>Definition</u> A glide reflection $^{\alpha}$ is a reflection over a line l composed with a translation τ_T parallel to l. $\alpha = \tau_T \circ \sigma_l$ (We require $T \neq \vec{0}$)

* Glide reflections are isometries - why?

<u>Exercise</u>: Draw enough pictures to convince yourself that if $T \parallel l$, then $\tau_T \circ \sigma_l = \sigma_l \circ \tau_T$.

<u>Corollary 4.35</u> A glide reflection has <u>no</u> fixed points.
(Draw enough pictures to convince yourself.)

We now know that an isometry falls into one of four categories:

1. no fixed points - translations and glide reflections - are these all?
2. exactly 1 fixed point - rotations around a point.
3. a line of fixed points, but not all points fixed - reflection over a line
4. all points fixed - identity.

In Section 4.10, we will see that this is a complete list of isometries.

Section 4.10 Classification of Isometries

Recall that every isometry can be written as the composition of 3 or fewer ~~isometries~~ reflections. This gives us another way to categorize isometries:

I. 1 reflection – of course this is a $\boxed{\text{reflection}}$

II. Composition of 2 reflections $\sigma_n \circ \sigma_m$

A. If $m \parallel n$, $\boxed{\text{translation}}$ (Thm. 4.24)

B. If m intersects n, $\boxed{\text{rotation}}$ (Thm. 4.28)

C. If $m = n$, $\boxed{\text{identity}}$

III Composition of 3 reflections $\sigma_n \circ \sigma_m \circ \sigma_\ell$

A. If $\ell \parallel m \parallel n$, $\boxed{\text{reflection}}$ (Thm. 4.26)

B. If ℓ, m, n concurrent, ~~rotation~~ reflection (Thm. 4.29)

C. If ℓ, m, n are neither concurrent nor all parallel (two of the three <u>may</u> be parallel) – ???

The next theorem completes this categorization.

<u>Theorem 4.37</u> If lines ℓ, m, n are neither concurrent nor all three parallel (two <u>may</u> be parallel), then

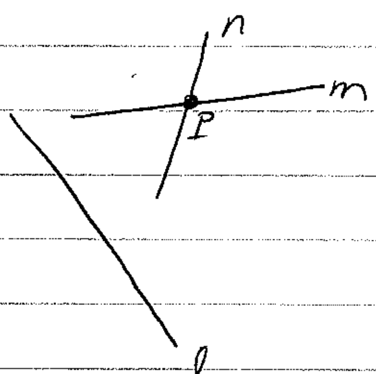
$\sigma_n \circ \sigma_m \circ \sigma_\ell$ is a glide reflection.

Proof of Theorem 4.37

To show that $\sigma_n \circ \sigma_m \circ \sigma_\ell$ is a glide reflection, we will find a line p and a vector $\vec{T}$ with $T \parallel p$, such that

$$\sigma_n \circ \sigma_m \circ \sigma_\ell = \underbrace{\tau_T \circ \sigma_p}_{\text{this is a glide reflection.}}$$

Since not all three lines are parallel, assume m and n intersect in a point P. Since we are assuming ℓ, m, n are not concurrent, ℓ does not go through P.



Let s be the perpendicular line to ℓ through P.

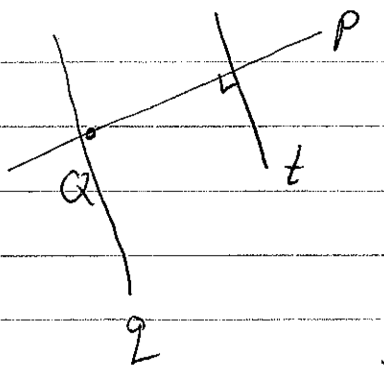
Then there is a line t, through P, so that $\sigma_n \circ \sigma_m = \sigma_t \circ \sigma_s$ (from ~~the proof~~ Theorem 4.28 - $\sigma_n \circ \sigma_m$ and $\sigma_t \circ \sigma_s$ are both rotations by θ degrees around P, where $\measuredangle(m,n) = \measuredangle(s,t) = \theta/2$).

Let Q be the intersection of s and ℓ (recall $s \perp \ell$) Then $\sigma_s \circ \sigma_\ell = \rho_{Q,180°}$, which is the central reflection σ_Q. Your textbook calls it σ_Q in this proof, but I will use the $\rho_{Q,180°}$ notation instead.

Proof of Theorem 4.37 cont'd

Now we can define the line p.

Let p be the perpendicular to t through Q. Define another line q as the parallel to t through Q.



Now we can also define the vector T: Since t and q are parallel, $\sigma_t \circ \sigma_q$ is a translation (Thm. 4.24); let T be the translation vector. $\sigma_t \circ \sigma_q = \tau_T$.

Now we will put this all together to show
$\sigma_n \circ \sigma_m \circ \sigma_\ell = \tau_T\, \sigma_p$.

$$\begin{aligned}
\sigma_n \circ \sigma_m \circ \sigma_\ell &= \sigma_t \circ \sigma_s \circ \sigma_\ell && (\text{Since } \sigma_n \circ \sigma_m = \sigma_t \circ \sigma_s) \\
&= \sigma_t \circ \rho_{Q,180°} && (\text{since } s \perp \ell, \text{ intersect at } Q) \\
&= \sigma_t \circ \sigma_q \circ \sigma_p && (\text{since } q \perp p, \text{ intersect at } Q) \\
&= \tau_T \circ \sigma_p && (\text{since } \sigma_t \circ \sigma_q = \tau_T).
\end{aligned}$$

Referring to Thm. 4.24 again, $T \perp t$ and $t \perp p$ by construction, so $T \parallel p$.
Therefore $\tau_T \circ \sigma_p$ is in fact a glide reflection,
so $\sigma_n \circ \sigma_m \circ \sigma_\ell$ is a glide reflection.