Math 403 Exam 2 Solutions

1. a) Orthocenter - the intersection of the altitudes of a triangle.

b) Isometry - a map from the plane to itself which preserves distances (i.e. $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ and for all X, Y, $d(\alpha(X), \alpha(Y)) = d(X, Y)$)

c) a linear isometry - an isometry α with the property that $\alpha(0) = 0$.

2. a) Any vectors X and Y which are perpendicular will work; for example, $X = (1,0)$ and $Y = (0,1)$.

b) No such vectors can exist since the Cauchy-Schwarz Inequality tells us $|X \cdot Y| \leq |X||Y|$.

c) Rotations and central reflections.

3. See page 70 of textbook.

4. (a) Let α be an isometry.
Define the vector A by $A = \alpha(0)$.
Define β by $\beta = \tau_{-A} \circ \alpha$

① β is an isometry because it is the composition of two isometries.

② $\beta(0) = \tau_{-A}(\alpha(0)) = \tau_{-A}(A) = A - A = 0$

③ $\tau_A \circ \beta = \tau_A \circ \tau_{-A} \circ \alpha = \iota \circ \alpha = \alpha$ $\square$.

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4(b) $A = \alpha(0) = -0 + 2C = 2(1,0) = (2,0)$

$\cancel{\beta(x)=}$ $\tau_{-A} \circ \alpha(x) = \tau_{-A}(-X+2C)$

$= \tau_{-A}(-X+(2,0)) = -X + (2,0) - (2,0) = -X$

$\boxed{\text{so } A = (2,0), \quad \beta(X) = -X}$

5. Let α be an isometry. By Problem 4, there exist a vector A and a linear isometry β such that $\alpha = \tau_A \circ \beta$.

Let l_{XY} be a line. Let $P \in l_{XY}$. Then $P = aX + bY$ where $a+b=1$.
Since β is linear,
$\beta(P) = \beta(aX+bY) = a\beta(X) + b\beta(Y)$, which is a point on $l_{\beta(X)\beta(Y)}$.

So β maps every point on l_{XY} to a point on $l_{\beta(X)\beta(Y)}$.

Since τ_A is a translation, it maps $l_{\beta(X)\beta(Y)}$ to a line (* see proof below). So $\alpha = \tau_A \circ \beta$ maps lines to lines.

(*) Let $Q \in l_{\beta(X)\beta(Y)}$. Then $Q = c\beta(X) + d\beta(Y)$ for some $c+d=1$. $\tau_A(Q) = c\beta(X) + d\beta(Y) + A$
$= c(\beta(X)+A) + d(\beta(Y)+A)$ since $c+d=1$.
So τ_A is a point on ~~$l_{\beta(X)}$~~ the line through $\beta(X)+A$ and $\beta(Y)+A$.

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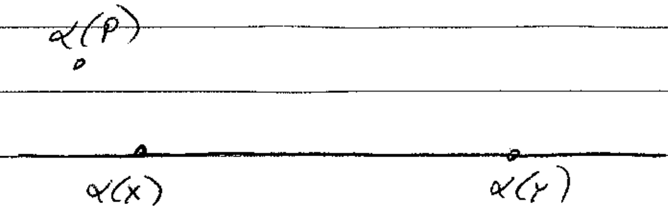
5. Alternate proof.

Let α be an isometry and let l_{XY} be a line. Let P be any point on l_{XY} between X and Y. We must show that $\alpha(P)$ is on $l_{\alpha(X)\alpha(Y)}$.

Suppose $\alpha(P)$ is not on $l_{\alpha(X)\alpha(Y)}$



Then, by the triangle inequality,

$$d(\alpha(X), \alpha(Y)) < d(\alpha(X), \alpha(P)) + d(\alpha(P), \alpha(Y))$$

Since α is isometry,

$$d(\alpha(X), \alpha(Y)) = d(X,Y),$$

$$d(\alpha(X), \alpha(P)) = d(X,P), \quad d(\alpha(P), \alpha(Y)) = d(P,Y).$$

So $d(X,Y) < d(X,P) + d(P,Y)$.

However, since P is on l_{XY} and between X and Y, $d(X,Y) = d(X,P) + d(P,Y)$.

This is a contradiction, so $\alpha(P)$ is on $l_{\alpha(X)\alpha(Y)}$. If P is on l_{XY} but not between X and Y, then rename the points and proceed as above.

6. a) T Proposition 4.10

b) T since $|V| = \sqrt{V \cdot V}$

c) T Corollary 3.9

d) F there are infinitely many such α.