**ROW & SEAT**

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Discussion Section      Family Name,    First Name      UIN

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# MATH 231 Section AL1

## TEST #3 YELLOW

April 21, 2008



## INSTRUCTIONS

Show ALL work on these pages. You may use the backside of the page if you need more room for your answer. You are allowed **50 MINUTES** to complete this test.

**Calculators are NOT permitted.**

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THIS PART IS FOR EXAMINER'S USE ONLY

| Question | Marks | Grade |
|---|---|---|
| 1 | 13 | |
| 2 | 6 | |
| 3 | 6 | |
| 4 | 10 | |
| 5 | 15 | |
| Total | 50 | |

**Problem 1.** Suppose the infinitely differentiable function $f(x)$ has

$$\sum_{k=0}^{\infty} e^k (x-3)^{2k}$$

as the Taylor series expanded around 3.

**(a)** State the values of $f'(3)$ and $f''(3)$. **[4 marks]**

Compare the first terms of $\sum_{k=0}^{\infty} e^k(x-3)^{2k}$ and $\sum_{n=0}^{\infty} \frac{f^{(n)}(3)}{n!}(x-3)^n$ to get

$$f'(3) = 0, \qquad f''(3) = 2!e^1.$$

**(b)** Determine the Taylor polynomials $P_2(x)$ and $P_4(x)$ for the given Taylor series. **[4 marks]**

Read off from the given Taylor series:

$$P_2(x) = 1 + e(x-3)^2, \qquad P_4(x) = 1 + e(x-3)^2 + e^2(x-3)^4.$$

**(c)** Find the radius of convergence and the interval of convergence of the given Taylor series. **[5 marks]**

Use the ratio test for the series $\sum_k a_k$ with $a_k = e^k(x-3)^{2k}$:

$$\left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{e^{k+1}(x-3)^{2k+2}}{e^k(x-3)^{2k}}\right| = e|x-3|^2 < 1$$

gives radius of convergence $R = e^{-1/2}$ and interval of convergence $(3 - e^{-1/2}, 3 + e^{-1/2})$.
[Note: The series $\sum_{k=0}^{\infty} e^k(x-3)^{2k}$ is clearly divergent at the endpoints $x = 3 \pm e^{-1/2}$.]

**Problem 2.** Use a known Taylor series to find the Taylor series for $f(x) = x^3e^{2x}$ expanded about $x = 0$. **[6 marks]**

We know the Taylor series $e^t = \sum_{k=0}^{\infty} \frac{t^k}{k!}$. Substitute $t = 2x$ and multiply by $x^3$ to obtain that $f(x)$ has the following Taylor series about 0:

$$x^3 \sum_{k=0}^{\infty} \frac{(2x)^k}{k!} = \sum_{k=0}^{\infty} \frac{2^k}{k!} x^{k+3}.$$

**Problem 3.** Use a known Taylor series to find the value of the limit **[6 marks]**

$$\lim_{x \to 0} \frac{e^{-x^4} - 1}{x^4}.$$

We know the Taylor series $e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots$. Substitute $t = -x^4$. So

$$\frac{e^{-x^4} - 1}{x^4} = \frac{-x^4 + \frac{x^8}{2!} - \frac{x^{12}}{3!} + \dots}{x^4} = \frac{-1}{1} + \frac{x^4}{2!} - \cdots \quad \xrightarrow{x \to 0} \quad -1$$

**Problem 4.** The *2-periodic even* function $f(x)$ is given on the interval $-1 \le x \le 1$ by the formula

$$f(x) = \begin{cases} x+1 & \text{if } -1 \le x < 0 \\ 1-x & \text{if } 0 \le x \le 1 \end{cases}$$

**(a)** Sketch the graph of $f(x)$ and display at least three periods. **[2 marks]**

(Graph of an even function with period 2)

**(b)** Determine all Fourier coefficients of the function $f(x)$. **[6 marks]**

The halfperiod of $f(x)$ is $L = 1$. We need to determine the $a_n$'s and $b_n$'s of the Fourier series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x) + b_n \sin(n\pi x). \qquad (*)$$

(I) $f(x)$ is even, so $\boxed{b_n = 0}$ for $n = 1, 2, \ldots$.

(II) From geometry of graph in part (a), we see $\boxed{a_0 = 1}$. [Or by computation, using $f(x)$ even, $a_0 = \int_{-1}^{1} f(x)dx = 2\int_0^1 f(x)dx = 2\int_0^1 (1-x)dx = 2\left[x - \frac{x^2}{2}\right]_{x=0}^{x=1} = 1.$]

(III) Again evenness of $f(x)$ gives

$$a_n = \int_{-1}^{1} f(x)\cos(n\pi x)dx = 2\int_0^1 f(x)\cos(n\pi x)dx = 2\int_0^1 (1-x)\cos(n\pi x)dx.$$

We compute $2\int_0^1 \cos(n\pi x)dx = 2\left[\frac{1}{n\pi}\sin(n\pi x)\right]_{x=0}^{x=1} = 0$ and, using integration by parts,

$$\begin{aligned} 2\int_0^1 x\cos(n\pi x)dx &= 2\left[x\frac{1}{n\pi}\sin(n\pi x)\right]_{x=0}^{x=1} - 2\int_0^1 1\frac{1}{n\pi}\sin(n\pi x)dx \\ &= 2\big(0-0\big) - \frac{2}{n\pi}\left[-\frac{1}{n\pi}\cos(n\pi x)\right]_{x=0}^{x=1} \\ &= \frac{2}{n^2\pi^2}\Big[\cos(n\pi x)\Big]_{x=0}^{x=1} = \frac{2}{n^2\pi^2}\Big(\cos(n\pi) - 1\Big) \end{aligned}$$

Altogether, $\boxed{a_n = -\frac{2}{n^2\pi^2}\Big(\cos(n\pi) - 1\Big) = \begin{cases} 0 & \text{for } n \text{ even} \\ \frac{4}{n^2\pi^2} & \text{for } n \text{ odd.} \end{cases}}$.

**(c)** State the Fourier series of $f(x)$ explicitely, using your results from part (b). **[2 marks]**

Plug framed results into $(*)$ to obtain the Fourier series

$$\frac{1}{2} + \sum_{n\,\text{odd}} \frac{4}{n^2\pi^2}\cos(n\pi x).$$

**Problem 5.** Let the closed curve $\mathcal{C}$ be given by the parametrization

$$\begin{aligned} x &= 3\sin(t) \\ y &= \cos(t) \end{aligned} \qquad \text{with } 0 \le t \le 2\pi.$$

**(a)** Eliminate the parameter $t$ to find an *x-y equation* of the curve $\mathcal{C}$. **[3 marks]**

Use $\cos^2(t) + \sin^2(t) = 1$ to obtain

$$\frac{1}{9}x^2 + y^2 = 1.$$

**(b)** Find the *slope* of the tangent line to the curve $\mathcal{C}$ at $t = \frac{\pi}{4}$. **[4 marks]**

Consider the derivatives $x' = 3\cos(t)$ and $y' = -\sin(t)$ at $t = \frac{\pi}{4}$, to obtain the slope

$$m = \frac{-\sin(\pi/4)}{3\cos(\pi/4)} = -\frac{1}{3}.$$

[Note that $\cos(\pi/4) = \sin(\pi/4)$ by geometry of unit circle!]

**(c)** Calculate the *area* enclosed by the curve $\mathcal{C}$. **[4 marks]**

Up to a sign coming from the orientation, the enclosed area is

$$\begin{aligned} \int_0^{2\pi} yx'dt &= \int_0^{2\pi} \cos(t)3\cos(t)\,dt \\ &= 3\int_0^{2\pi} \cos^2 t\,dt = 3\int_0^{2\pi} \frac{1}{2}(1+\cos 2t) \\ &= 3\pi + \frac{3}{2}\int_0^{2\pi} \cos 2t\,dt = 3\pi. \end{aligned}$$

Thus the enclosed area is $3\pi$.

**(d)** Set up, *but do not evaluate*, the integral for the *arc length* of the curve $\mathcal{C}$. **[4 marks]**

Using the derivatives calculated in part (b),

$$L = \int_0^{2\pi} \sqrt{(x')^2 + (y')^2}\,dt = \int_0^{2\pi} \sqrt{9\cos^2(t) + \sin^2(t)}\,dt$$