An Ore-type Theorem on Equitable Coloring

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December 31, 2006

Abstract

A proper vertex coloring of a graph is *equitable* if the sizes of its color classes differ by at most one. In this note, we prove that if G is a graph such that for each edge $xy \in E(G)$, the sum $d(x) + d(y)$ of the degrees of its ends is at most $2r + 1$ then G has an equitable coloring with $r + 1$ colors. This extends the Hajnal-Szemerédi Theorem on graphs with maximum degree r and a recent conjecture by Kostochka and Yu. We also pose an Ore-type version of the Chen-Lih-Wu Conjecture and prove a very partial case of it.

1 Introduction

An *equitable k-coloring* of a graph G is a proper k-coloring, for which any two color classes differ in size by at most one. It can be viewed as a packing of G with the $|V(G)|$-vertex graph, whose components are cliques with either $\lfloor |V(G)|/k \rfloor$ or $\lceil |V(G)|/k \rceil$ vertices. Recall that two n-vertex graphs *pack*, if there exists an edge disjoint placement of these graphs into K_n. In other words, G_1 and G_2 *pack* if G_1 is isomorphic to a subgraph of the complement of G_2 (and vice versa). A number of important graph theoretic problems can be naturally expressed in the language of packing. For example, the classical Dirac's Theorem [5] on the existence of hamiltonian cycles in each n-vertex graph with minimum degree at least $n/2$ can be stated in terms of packing as follows: *Let $n \geq 3$. If G is an n-vertex graph and its maximum degree, $\Delta(G)$, is at most $\frac{1}{2}n - 1$, then G packs with the cycle C_n of length n.*

Similarly, Ore's theorem [14] on hamiltonian cycles is as follows: *If $n \geq 3$ and G is an n-vertex graph with $d(x) + d(y) \leq n - 2$ for each edge $xy \in E(G)$, then G packs with the cycle C_n.*

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One of the main known results on equitable coloring is the Hajnal-Szemerédi Theorem [7] stating that every graph with maximum degree $\Delta(G) \le r$ has an equitable $(r+1)$-coloring. It has many applications. Alon and Füredi [1], Alon and Yuster [2, 3], Janson and Ruciński [8], Pemmaraju [15] and Rödl and Ruciński [16] used this theorem to derive bounds for sums of dependent random variables with limited dependence or to prove the existence of some special vertex partitions of graphs and hypergraphs. We call the Hajnal-Szemerédi Theorem a *Dirac-type* result, since it provides a packing of a graph G with a special graph given a restriction on the maximum degree of G. Recently, Kostochka and Yu [11, 12] conjectured that the following Ore-type result holds true: *Every graph in which $d(x) + d(y) \le 2r$ for every edge xy has an equitable $(r + 1)$-coloring.* Clearly, this conjecture implies the Hajnal-Szemerédi Theorem. In this note, we prove the following somewhat stronger result.

Theorem 1 *Every graph satisfying $d(x) + d(y) \le 2r + 1$ for every edge xy, has an equitable $(r + 1)$-coloring.*

The proof elaborates the ideas of the original proof of the Hajnal-Szemerédi Theorem [7] and of the recent short proof of it in [9]. There are more graphs for which Theorem 1 is tight, than those for which the Hajnal-Szemerédi Theorem is tight. For example, for each odd $m \le r + 1$, the graph $K_{m,2r+2-m}$ satisfies the condition $d(x) + d(y) \le 2r + 2$ for every edge xy and has no equitable $(r + 1)$-coloring. We conjecture that the following Ore-type analogue of the Chen-Lih-Wu Conjecture holds.

Conjecture 2 *Let $r \ge 3$. If G is a graph in which $d(x) + d(y) \le 2r$ for every edge xy and G has no equitable r-coloring, then G contains either K_{r+1} or $K_{m,2r-m}$ for some odd m.*

We also prove that Conjecture 2 holds for $r = 3$.

The structure of the text is as follows. In the next section we prove Theorem 1. The key ingredients of the proof are a recoloring lemma and a discharging proof of the nonexistence of a bad example. In the last section we describe and discuss the Chen-Lih-Wu Conjecture and its extension, Conjecture 2.

Most of our notation is standard; possible exceptions include the following. For a graph G, we let $|G| := |V(G)|$, $\|G\| := |E(G)|$ and $\bar{\sigma}(G) := \max\{d(x) + d(y) : xy \in E(G)\}$. For a vertex y and set of vertices X, $N_X(y) := N(y) \cap X$ and $d_X(y) := |N_X(y)|$. If μ is a function on edges then $\mu(A, B) := \sum_{xy \in E(A,B)} \mu(x, y)$, where $E(A, B)$ is the set of edges linking a vertex in A to a vertex in B. For a set S and element x we write $S + x$ for $S \cup \{x\}$ and $S - x$ for $S \setminus \{x\}$. For a function $f : V \to Z$, the restriction of f to $W \subseteq V$ is denoted by $f|W$. Functions are viewed formally as sets of ordered pairs.

2 Main proof

In this section we prove Theorem 1. Let G be a graph satisfying $\bar{\sigma}(G) \le 2r + 1$. We may assume that $|G|$ is divisible by $r + 1$. To see this, suppose that $|G| = s(r + 1) - p$, where $p \in [r]$. Let $G' := G + K^p$. Then $|G'|$ is divisible by $r + 1$ and $\Delta(G') \le r$. Moreover, the restriction of any equitable $(r+1)$-coloring of G' to G is an equitable $(r+1)$-coloring of G. So let $|G| = rs$.

Suppose for a contradiction, that G is an edge-minimal counterexample to the theorem. Consider any edge $e = xy$ with $d(x) \leq d(y)$. By minimality, there exists an equitable $(r+1)$-coloring of $G - e$. Since G is a counterexample, some color class V contains both x and y. Since $\bar{\sigma}(G) \leq 2r+1$, $d(x) \leq r$. Thus there exists a class W such that x has no neighbors in W. Moving x to W yields an $(r+1)$-coloring f of G with all classes of size s, except for one *small* class $V^-(f) = V - x$ of size $s-1$ and one *large* class $V^+(f) = W + x$ of size $s+1$. We say that such a coloring is *nearly equitable.*

Given a coloring f with a unique small class V^- (but possibly no large class), define an auxiliary digraph $\mathcal{H} = \mathcal{H}(f)$ as follows. The vertices of $\mathcal{H}$ are the color classes of f. A directed edge UW belongs to $E(\mathcal{H})$ iff some vertex $y \in U$ has no neighbors in W. In this case we say that y is *movable* to W. Call $W \in V(\mathcal{H})$ *accessible*, if V^- is reachable from W in $\mathcal{H}$. So V^- is trivially accessible.

Lemma 3 *If G has a nearly equitable coloring, whose large class V^+ is accessible, then it has an equitable coloring with the same number of colors.*

Proof. Let $\mathcal{P} = V_1, \ldots, V_k$ be a path in $\mathcal{H}$ from $V_1 := V^+$ to $V_k := V^-$. This means that for each $j = 1, \ldots, k-1$, V_j contains a vertex y_j that has no neighbors in V_{j+1}. So, if we move y_j to V_{j+1} for $j = 1, \ldots, k-1$, then we obtain an equitable coloring with the same number of color classes. ■

Let $\mathcal{A} = \mathcal{A}(f)$ denote the family of accessible classes and $\mathcal{B}$ denote the family of non-accessible classes. Then $V^- \in \mathcal{A}$ and, by Lemma 3, $V^+ \in \mathcal{B}$. Set $A := \bigcup \mathcal{A}$, $B := \bigcup \mathcal{B}$, $m := |\mathcal{A}| - 1$ and $q := |\mathcal{B}| = r - m$. Then $|A| = (m+1)s - 1$ and $|B| = qs + 1$.

Each vertex $y \in B$ has a neighbor in each class of A and so satisfies $d_A(y) \geq m+1$. (1)

It follows that

$$\bar{\sigma}(G[B]) \leq \bar{\sigma}(G) - 2(m+1) \leq 2q - 1.$$

Thus by the minimality of G,

Every subgraph of $G[B]$ has an equitable q-coloring. (2)

For an accessible class $U \in \mathcal{A}(f)$, define $\mathcal{S}_U := \mathcal{S}_U(f)$ to be the set of classes $X \in \mathcal{A}$ such that there is an $X - V^-$ path in $\mathcal{H}(f) - U$ and $\mathcal{T}_U := \mathcal{T}_U(f) := \mathcal{A} \setminus (\mathcal{S}_U + U)$. Call U *terminal*, if $\mathcal{S}_U = \mathcal{A} - U$; otherwise U is non-terminal. Note that if U is non-terminal then $\mathcal{T}_U \neq \emptyset$. Trivially, V^- is non-terminal unless $m = 1$, in which case it is terminal. Let $W \in \mathcal{A}$ be terminal. An edge zy with $z \in W$ and $y \in B$, is *solo* if $N_W(y) = \{z\}$. The ends of solo edges are called *solo vertices* and vertices linked by solo edges are called *special neighbors* of each other.

Lemma 4 *Suppose that $W \in \mathcal{A}$ is terminal. If $z \in W$ is solo then z has a neighbor in every class of $\mathcal{A} - W$. In particular $d_A(z) \geq m$.*

Proof. Suppose for a contradiction that z has a special neighbor $y \in B$ and no neighbor in $X \in \mathcal{A} - W$. Since W is terminal there exists a path $\mathcal{P}$ from X to V^- in $\mathcal{H} - W$. Move z to X and y to W. By hypothesis $X^* := X + z$ is independent and, since xy is solo, $W^* := W + y - z$ is independent. This yields a nearly equitable coloring f^* of $G[A+y]$ with $V^+(f^*) = X + z$. Moreover $\mathcal{P}^* := \mathcal{P} + V^+(f^*) - X$ is a path from $V^+(f^*)$ to $V^-(f^*)$ in $\mathcal{H}(f^*)$. By Lemma 3, $G[A+y]$ has an equitable $(m+1)$-coloring h_1. By (2), $G[B] - y$ has an equitable q-coloring h_2. Thus $h_1 \cup h_2$ is an equitable $(r+1)$-coloring of G, a contradiction. ■

Define a non-empty family $\mathcal{A}' := \mathcal{A}'(f) \subseteq \mathcal{A}(f)$ as follows. If $m = 0$ then set $\mathcal{A}' := \mathcal{A}$. Otherwise, V^- is a non-terminal class, and so such classes exist. Choose a non-terminal U so that $|\mathcal{T}_U|$ is minimum and set $\mathcal{A}' := \mathcal{T}_U$. Let $A' := A'(f) := \bigcup \mathcal{A}'$ and $t := t(f) := |A'|$.

Lemma 5 *The family $\mathcal{A}'$ satisfies the following:*
(P1) Every class in $\mathcal{A}'$ is terminal.
(P2) $d_A(x) \geq m - t$ for all $x \in A'$.

Proof. If $m = 0$ then the only accessible class V^- is terminal. So $\mathcal{A}' = \mathcal{A}$ satisfies (P1) and (P2) trivially. Otherwise $m > 0$ and $\mathcal{A}' = \mathcal{T}_U$ for some non-terminal $U \in \mathcal{A}$. Consider $X \in \mathcal{T}_U$. Then $\mathcal{T}_X \subseteq \mathcal{T}_U$. By the minimality of $\mathcal{T}_U$, X is terminal. So (P1) holds true.

No class in $\mathcal{A}' = \mathcal{T}_U$ has an outneighbor in $\mathcal{S}_U$. It follows that every vertex in A' has a neighbor in each of the $m - t$ classes of $\mathcal{S}_U$. So (P2) holds true. ■

Define an *obstruction* to be a nearly equitable $(r+1)$-coloring f such that
(C1) $m(f) = |\mathcal{A}(f)|$ is maximum; and
(C2) subject to (C1), $t(f) = |\mathcal{A}'(f)|$ is minimum.

Lemma 6 *Suppose that f is an obstruction, $W \in \mathcal{A}'$ and $z \in W$ is a solo vertex with a special neighbor $y \in B$. Set $A^- := A - z$. Then G has an obstruction g such that*

$$g|A^- = f|A^- \text{ and } g(y) = f(z). \tag{3}$$

Proof. Set $W^* := W + y - z$. Since zy is a solo edge, W^* is independent. Thus switching y and z yields an equitable $(m+1)$-coloring h_1 of $G[A^*]$, where $A^* := A + y - z$. Our plan is to extend h_1 to an obstruction. Any such extension will satisfy (3). For this we will need the following analysis of $\mathcal{H}(h_1)$.

Set $\mathcal{H}_0 := \mathcal{H}(f)[\mathcal{A}(f)]$. For $X \in \mathcal{A}$, let $X^* := X$, if $X \neq W$; otherwise let $X^* := W^*$. Then $\mathcal{H}_0 - W = \mathcal{H}(h_1) - W^*$. Moreover, by (1) and Lemma 4, neither y nor z is movable to any class in $\mathcal{H}_0 - W$. It follows that the outneighborhood of W in $\mathcal{H}_0$ is the same as the outneighborhood of W^* in $\mathcal{H}(f)$. In other words,

$$* : \mathcal{H}_0 - E^-(W) \to \mathcal{H}(h_1) - E^-(W^*)$$

is an isomorphism. Let $\mathcal{P} := X_1 \dots X_t$ and $\mathcal{P}^* := X_1^* \dots X_t^*$ be the image of $\mathcal{P}$. Then $\mathcal{P}$ is a path in $\mathcal{H}_0$ with $W \notin V(\mathcal{P}) - X_1$ iff $\mathcal{P}^*$ is a path in $\mathcal{H}_1$ with $W^* \notin V(\mathcal{P}^*) - X_1^*$. Since W is terminal by (P1), it follows that every class of h_1 is accessible in $\mathcal{H}(h_1)$, i.e. $\mathcal{A}^*(f) = \mathcal{A}(h_1)$, where $\mathcal{A}^*(f)$ is the image of $\mathcal{A}(f)$.

Set $B^- := B - y$. By (2) $G[B^-]$ has an equitable q-coloring h_2. Using that W is terminal, Lemma 4 and (1), we have

$$2r + 1 \geq d(z) + d(y) = d_A(z) + d_A(y) + d_B(z) + d_B(y) \geq 2m + 1 + d_B(z) + d_B(y).$$

In other words,

$$2q \geq d_B(z) + d_B(y).$$

Since z is adjacent to y, $d_{A^*}(z) \geq d_A(z) + 1 = m + 1$ and $d_{B^-}(z) \leq 2q - 1$. If there exists a class $X \subseteq B^-$ of h_2 such that z has no neighbors in X then move z to X to obtain a q-coloring h_3 of $G[B^*]$, where $B^* := B^- + z$. Then $g := h_1 \cup h_3$ is a nearly equitable $(r+1)$-coloring of G. Otherwise $d_{B^-}(z) \geq q$ and $d(z) \geq q + m + 1 = r + 1$. Since $d_{B^-}(z) \leq 2q - 1$, some class X of h_2 contains exactly one neighbor w of z. Switch z and w to obtain a q-coloring h_4 of $G[B^*] - w$. Then $f' = h_1 \cup h_4$ is an equitable coloring of $G - w$ with one small class V^- and no large class. Since $d(z) \geq r + 1$ and z is adjacent to w, $d(w) \leq r$. It follows that w can be added to some class of f', yielding a large class.

First suppose that w can be added to $X^* \subseteq A^*$. This yields a nearly equitable coloring h' of $A^* + w$ with large class $X^* + w$. Since $X^* \in \mathcal{A}(h_1)$, and $X^* + w \in \mathcal{A}(h')$. By Lemma 3, there exist a nearly equitable $(m+1)$-coloring h'' of $G[A^* + w]$. Then $h'' \cup h_4$ is an equitable $(r+1)$-coloring of G, a contradiction. So w can be moved to some $X \subseteq B^*$. Let g be the nearly equitable $(r+1)$-coloring obtained from $h_1 \cup h_4$ by moving w to X. Regardless of the case, we have constructed a nearly equitable $(r+1)$-coloring g that satisfies (3). We still must show that g satisfies (C1) and (C2).

First, we show that g satisfies (C1). Since f satisfies (C1) it suffices to show that $m(f) \leq m(g)$, which follows from $\mathcal{A}^*(f) = \mathcal{A}(h_1) \subseteq \mathcal{A}(g)$. So g satisfies (C1) and $\mathcal{A}(h_1) = \mathcal{A}(g)$Now we show that g satisfies (C2). Suppose that $\mathcal{A}'(f) = \mathcal{T}_U$, where U is non-terminal in $\mathcal{H}(f)$. Since f satisfies (C2), it suffices to show that $t(g) \leq t(f)$. We will do this by showing that $W^* \in \mathcal{T}_U(g)$ and $\mathcal{S}_U(f) \subseteq \mathcal{S}_U(g)$. Then U is non-terminal in $\mathcal{H}(g)$ and $t(g) \leq |\mathcal{T}_U(g)| \leq |\mathcal{T}_U(f)| = t(f)$. Suppose that $\mathcal{P}^*$ is a $W^* - V^-$ path in $\mathcal{H}_g$. Then $\mathcal{P}^* \subseteq \mathcal{A}(g) = \mathcal{A}(h_1)$. So its inverse $\mathcal{P}$ under * is a $W - V^-$ path in $\mathcal{H}(f)$. Since $W \in \mathcal{T}_U$, U must be a vertex of $\mathcal{P}$ and thus $\mathcal{P}^*$. So $W^* \in \mathcal{T}_U(g)$. Now suppose that $X \in \mathcal{S}_U(f)$. Then there exists an $X - V^-$ path $\mathcal{P}$ in $\mathcal{H}(f) - U$. It follows that $\mathcal{P}^*$ is an $\mathcal{H}(h_1) - U \subseteq \mathcal{H}(g) - U$ path and so $X \in \mathcal{S}_U(g)$. So (C2) holds and g is an obstruction. ■

Suppose that f is an obstruction and $z \in A'$ is a solo vertex with a special neighbor $y \in B$. Let S^y be the set of special neighbors of y in A'. By (P2), y has a neighbor in every class of $\mathcal{A}$; moreover if $W \in \mathcal{A}'$ and y does not have a solo neighbor in W then y has at least two neighbors in W. Thus

$$d_{A'}(y) \geq 2t - |S^y| \text{ and } d_A(y) \geq m + 1 + t - |S^y|. \tag{4}$$

Set $c_y := \max\{d_B(z) : z \in S^y\}$ if $S^y \neq \emptyset$; otherwise $c_y := 1$. Similarly, set $c'_y := \max\{d_B(z) : z \in N_{A'}(y) \setminus S^y\}$ if $N_{A'}(y) \neq S^y$; otherwise $c'_y := 1$. Define a weight function μ on $E(A', B)$

by

$$\mu(xy) := \frac{q}{d_B(x)}.$$

We shall finish our proof by proving the following three contradictory claims.

Claim 7 *For all obstructions f, there exists a vertex $y \in B$ such that $\mu(A', y) < t$.*

Claim 8 *For all obstructions f and all vertices $y \in B$, if $\mu(A', y) < t$ then y is solo. Moreover, in this case, either $c_y \geq q + 1$ or $c'_y \geq 2q + 1$.*

Claim 9 *There exists an obstruction f such that $\mu(A', y) \geq t$ for all solo vertices $y \in B$.*

Proof of Claim 7. For any $x \in A$, if $N_B(x) \neq \emptyset$ then

$$\mu(x, B) = \sum_{y \in N_B(x)} \frac{q}{d_B(x)} = q;$$

otherwise $\mu(x, B) = 0$. Regardless,

$$\mu(x, B) \leq q.$$

Thus

$$qst \geq q\,|A'| \geq \sum_{x \in A'} \mu(x, B) = \mu(A', B) = \sum_{y \in B} \mu(A', y) \geq |B| \min_{y \in B} \mu(A', y) > qs \min_{y \in B} \mu(A', y)$$

and so $\mu(A', y) < t$ for some $y \in B$. ■

Proof of Claim 8. Let $\mu(A', y) < t$. Let $\mathcal{S} := \{W \in \mathcal{A}' : S^y \cap W \neq \emptyset\}$ and $\mathcal{D} := \mathcal{A}' \setminus \mathcal{S}$. First suppose that $c'_y \leq 2q$. Then

$$t > \mu(A', y) = \sum_{W \in \mathcal{S}} \sum_{x \in N_W(y)} \frac{q}{d_B(x)} + \sum_{W \in \mathcal{D}} \sum_{x \in N_W(y)} \frac{q}{d_B(x)} \geq |\mathcal{S}| \frac{q}{c_y} + 2\,|\mathcal{D}| \frac{q}{c'_y} = |\mathcal{S}| \frac{q}{c_y} + |\mathcal{D}|.$$

Thus $|\mathcal{D}| < t$ and so $|S^y| = |\mathcal{S}| > 0$. Thus y is solo. Moreover, $\frac{q}{c_y} < 1$ and so $c_y \geq q + 1$.

Now suppose that $d_B(x) \geq 2q + 1$ for some $x \in N_{A'}(y)$. Using (P2) and (4),

$$\begin{aligned}
2r + 1 &\geq d(x) + d(y) = d_A(x) + d_B(x) + d_A(y) + d_B(y) \\
&\geq (m - t) + (2q + 1) + (m + 1 + t - |S^y|) \\
&= 2(m + q + 1) - |S^y| \\
&= 2r + 2 - |S^y|
\end{aligned}$$

It follows that $|S^y| \geq 1$ and so y is again solo. ■

Proof of Claim 9. CASE 1: $t \geq q$. Choose an obstruction f such that $|E(A', B)|$ is minimum. Let $y \in B$ be solo and $z \in S^y$. Let g be an obstruction satisfying the conclusion of Lemma 6. Set $A^- := A' - z$ and $B^- := B - y$. By the choice of f, $|E(A'(f), B(f))| \leq$

$|E(A'(g), B(g))|$ and so $d_{A^-}(y) + d_{B^-}(z) \leq d_{A^-}(z) + d_{B^-}(y)$. Recalling that y is adjacent to z and using (1) and Lemma 4,

$$\begin{aligned} d_{A'}(y) + d_B(z) &\leq \left\lfloor \frac{(d_{A^-}(y) + d_{B^-}(y) + d_{A^-}(z) + d_{B^-}(z))}{2} \right\rfloor + 2 \qquad (5) \\ &\leq \left\lfloor \frac{d(y) - (m+2-t) + d(z) - (m+2-t)}{2} \right\rfloor + 2 \\ &\leq \left\lfloor \frac{2r+1-2m+2t}{2} \right\rfloor \\ &= t + q. \end{aligned}$$

In particular, since $d_{A'}(y) \geq t$ by (1), $d_B(z) \leq q$. So $c_y \leq q$, since z was an arbitrary special neighbor of y. By (4) and (5),

$$|S^y| \geq 2t - d_{A'}(y) \geq 2t - (t + q - d_B(z)) \geq t - q + c_y.$$

So

$$\mu(A', y) \geq \sum_{z \in S^y} \frac{q}{d_B(z)} \geq |S^y| \frac{q}{c_y} \geq (t - q + c_y) \frac{q}{c_y} = (t-q)\frac{q}{c_y} + q \geq t.$$

CASE 2: $q \geq t$. Choose an obstruction f such that $\|G[B]\|$ is as large as possible. Then $d_B(z) \leq d_B(y) + 1$ for all solo edges zy with $z \in A'$. Thus, using Lemma 4 and (1)

$$\begin{aligned} 2r + 1 &\geq d_A(z) + d_B(z) + d_A(y) + d_B(y) \geq 2m + 1 + d_B(z) + d_B(y) \\ 2q &\geq d_B(z) + d_B(y) \geq 2d_B(z) - 1 \\ q &\geq d_B(z). \end{aligned}$$

Since z was arbitrary, $c_y \leq q$. If $\mu(A', y) < t$, then, by Claim 8, y has a neighbor x such that $d_B(x) \geq 2q + 1$. Moreover $d_B(y) \geq c_y - 1$. So, using (P2), (1) and (4),

$$2r+1 \geq d(x) + d(y) \geq (m - t + 2q + 1) + (m + 1 - t + 2t - |S^y| + c_y - 1) = 2r + 1 - |S^y| + c_y.$$

Thus $|S^y| \geq c_y$. So $\mu(A', y) \geq |S^y| \frac{q}{c_y} \geq q \geq t$. ■

Since Claims 7-9 are contradictory, this completes the proof of the theorem.

3 On two conjectures

Chen, Lih and Wu [4] proposed the following analogue of Brooks' Theorem for equitable coloring:

Conjecture 10 *[4] Let G be a connected graph with maximum degree Δ. If G is distinct from $K_{\Delta+1}$, $K_{\Delta,\Delta}$, and is not an odd cycle, then G has an equitable Δ-coloring.*

This conjecture is proved for some classes of graphs such as bipartite graphs [13], outerplanar graphs [17], planar graphs with maximum degree at least 13 [18], graphs with the average degree five times less than the maximum degree [10] and others. In particular, Chen, Lih and Wu [4] proved that the conjecture holds for $\Delta = 3$:

Theorem 11 *If G is a connected graph with $\Delta(G) \leq 3$ distinct from K_4 and $K_{3,3}$, then G has an equitable 3-coloring.*

If we consider Ore-type setting, then for every odd $m \leq r$, the graph $G_{r,m} = K_{m,2r-m}$ has $\overline{\sigma}(G_{r,m}) = 2r$ and has no equitable r-coloring. However, we believe that Conjecture 2 stated in the introduction holds true. To support the conjecture, we prove that it is true for $r = 3$. Note that the word 'connected' is not present in the statement, but this is an equivalent form.

Theorem 12 *If G is a graph with $d(x) + d(y) \leq 6$ for each $xy \in E(G)$ and if G does not contain any of the graphs K_4, $K_{3,3}$ and $K_{5,1}$, then G has an equitable 3-coloring.*

Proof. Let G be an edge-minimal counterexample to the theorem. Let v be a vertex of the maximum degree in G. If $d(v) = 5$, then G contains $K_{5,1}$, a contradiction to our assumption. By Theorem 11, $d(v) > 3$. Hence $d(v) = 4$. Let w_1, w_2, w_3, w_4 be the neighbors of v. Under the constrains on the graph, $d(w_i) \leq 2$ for each $i = 1, 2, 3, 4$. For $i = 1, 2, 3, 4$, let u_i be the neighbor of w_i distinct from v, if it exists.

CASE 1: u_1 does not exist or $u_1 = w_2$. Consider $G' = G - v - w_1 - w_2$. Since G' is a proper subgraph of G, it satisfies the conditions of the theorem. By the minimality of G, there exists an equitable 3-coloring f of G'. We extend f to an equitable 3-coloring of G as follows: Choose a color $\alpha \in \{1, 2, 3\} - f(w_3) - f(w_4)$ as $f(v)$, then choose a color $\beta \in \{1, 2, 3\} - f(u_2) - \alpha$ as $f(w_2)$, and finally choose the color $\gamma \in \{1, 2, 3\} - \alpha - \beta$ as $f(w_1)$.

So, below all u_i exist and are distinct from all w_j.

CASE 2: $u_3 = u_4$. Consider $G'' = G - \{v, w_1, w_2, w_3, w_4, u_3\}$. By the minimality of G, there exists an equitable 3-coloring f of G''. We extend f to the whole G as follows. First assign to u_3 and v a color α distinct from the colors of neighbors of u_3 in G'' (there are at most two such neighbors). Then for $i = 1, 2$, let $f(w_i) \in \{1, 2, 3\} - f(u_i) - \alpha$. Finally, for $i = 3, 4$, let $f(w_i) \in \{1, 2, 3\} - f(w_{i-2}) - \alpha$. Since each color appears exactly twice on $\{v, w_1, w_2, w_3, w_4, u_3\}$, we have an equitable 3-coloring of G.

Thus below all u_i are distinct and the only remaining case is as follows.

CASE 3: All u_i exist and are distinct; furthermore the set $\{w_1, w_2, w_3, w_4\}$ is independent. Let G''' be the graph obtained from $G - v$ by merging w_1 with w_3 into a new vertex w_1^* and merging w_2 with w_4 into a new vertex w_2^*. Since the two new vertices have degree exactly 2 each, G''' does not contain any of K_4, $K_{3,3}$ and $K_{5,1}$. Hence there exists an equitable 3-coloring f of G'''. We may assume that $f(w_1^*) = 1$. If $f(w_2^*) \neq 1$, then we may assume that $f(w_2^*) = 2$ and let $f(w_1) = f(w_3) = 1$, $f(w_2) = f(w_4) = 2$, and $f(v) = 3$.

Suppose that $f(w_1^*) = f(w_2^*) = 1$. We may assume that $f(u_4) = 2$. Then we let $f(w_1) = f(w_2) = f(w_3) = 1$, $f(w_4) = 3$, and $f(v) = 2$.

Thus in all cases we find an equitable 3-coloring of G, a contradiction. ■

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