

Coloring uniform simple hypergraphs with few edges

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Based on joint work with **Mohit Kumbhat**

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Let $m(r, k, g)$ denote the minimum number of edges in an **r -uniform non- k -colorable** hypergraph of girth g , and $f(r, k, b)$ denote the minimum number of edges in an **r -uniform non- k -colorable b -simple** hypergraph.

Theorem 1. [Erdős and Lovász] If G is an r -uniform hypergraph and $\Delta(G) \leq k^{r-1}/4r$, then G is k -colorable.

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$$m(r, k, 3) = f(r, k, 1) \geq \frac{k^{2(r-2)}}{16r(r-1)^2}.$$

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Theorem 3. [Erdős and Lovász]

$$m(r, k, g) \leq 4 \cdot 20^{g-1} r^{3g-5} k^{(g-1)(r+1)}.$$

Deriving Cor. 2 from Th. 1

Suppose H is an r -uniform non- k -colorable simple hypergraph. For each $e \in E(H)$, let $v(e)$ be a vertex in e of max. degree in H .

Let $e' = e - v(e)$ and $E(H') = \{e' : e \in H\}$.

By definition, H' is $(r - 1)$ -uniform non- k -colorable and simple. By

Th. 1, H' has a vertex w incident with $d = k^{r-1}/4r$ edges

e'_1, \dots, e'_d . Hence each of the corresponding edges e_i ,

$i = 1, \dots, d$ in H contains a vertex $v(e_i) \neq w$ of degree at least

d . Since H is simple, all $v(e_i)$ are distinct. Thus

$$\sum_{v \in V(H)} \deg_H(v) \geq d^2. \Rightarrow e(H) \geq d^2/r.$$

Theorem 4. [Z. Szabó] For every $\epsilon > 0$ and positive integer k , there exists an $r_0 = r_0(\epsilon, k)$ such that for every $r > r_0$, every r -uniform simple hypergraph G with $\Delta(G) \leq k^r r^{-\epsilon}$ is k -colorable.

Corollary 5. [Z. Szabó] For every $\epsilon > 0$ and positive integer k , there exists an $r_0 = r_0(\epsilon, k)$ such that for every $r > r_0$,

$$m(r, k, 3) = f(r, k, 1) \geq \frac{k^{2r}}{r^{1+2\epsilon}}.$$

For $0 < \epsilon < 1$ and positive integer k , an edge e of an r -uniform hypergraph G is (ϵ, k) -heavy, if at least $r/2$ vertices of e have degree at least $k^r r^{-\epsilon}$ in G . If an edge is not (ϵ, k) -heavy, then it is (ϵ, k) -light.

Observation 6. Suppose that an r -uniform hypergraph G has a vertex contained in at least $k^r r^{-\epsilon}$ (ϵ, k) -heavy edges. If G is simple, then $|E(G)| \geq 0.25k^{2r} r^{-2\epsilon}$.
If in addition, G has no 3-cycles, then $|E(G)| \geq 0.25k^{2r} r^{1-2\epsilon}$.

Main result

Theorem 7. [Kostochka and Kumbhat] For every $\epsilon > 0$ and positive integers k and b , there is an $r_0 = r_0(\epsilon, k, b)$ s. t. for every

$r > r_0$, every r -uniform b -simple hypergraph G satisfying

(a) $\Delta(G) \leq k^{r+0.5} r^{-\epsilon}$ and

(b) every vertex of G is contained in less than $k^r r^{-\epsilon}$ (ϵ, k) -heavy edges

is k -colorable.

Corollaries

Corollary 8. [K. and K.] For every $\epsilon > 0$ and positive integers k and b , there exists an $r_0 = r_0(\epsilon, k, b)$ such that for every $r > r_0$,

$$k^{(b+1)r/b} r^{-\epsilon} \leq f(r, k, b) \leq 10k^2 (2k^r r^2)^{(b+1)/b}.$$

Corollary 9. [K. and K.] For every $\epsilon > 0$ and positive integers k and g , there exists an $r_0 = r_0(\epsilon, k, g)$ such that for every $r > r_0$,

$$m(r, k, g) \geq k^{gr/2} r^{-\epsilon}.$$

Compare with $m(r, k, g) \leq 4 \cdot 20^{g-1} r^{3g-5} k^{(g-1)(r+1)}$. [E. & L.]