Ore-type graph coloring problems

Alexandr Kostochka

Based on joint work with H. Kierstead

Definition

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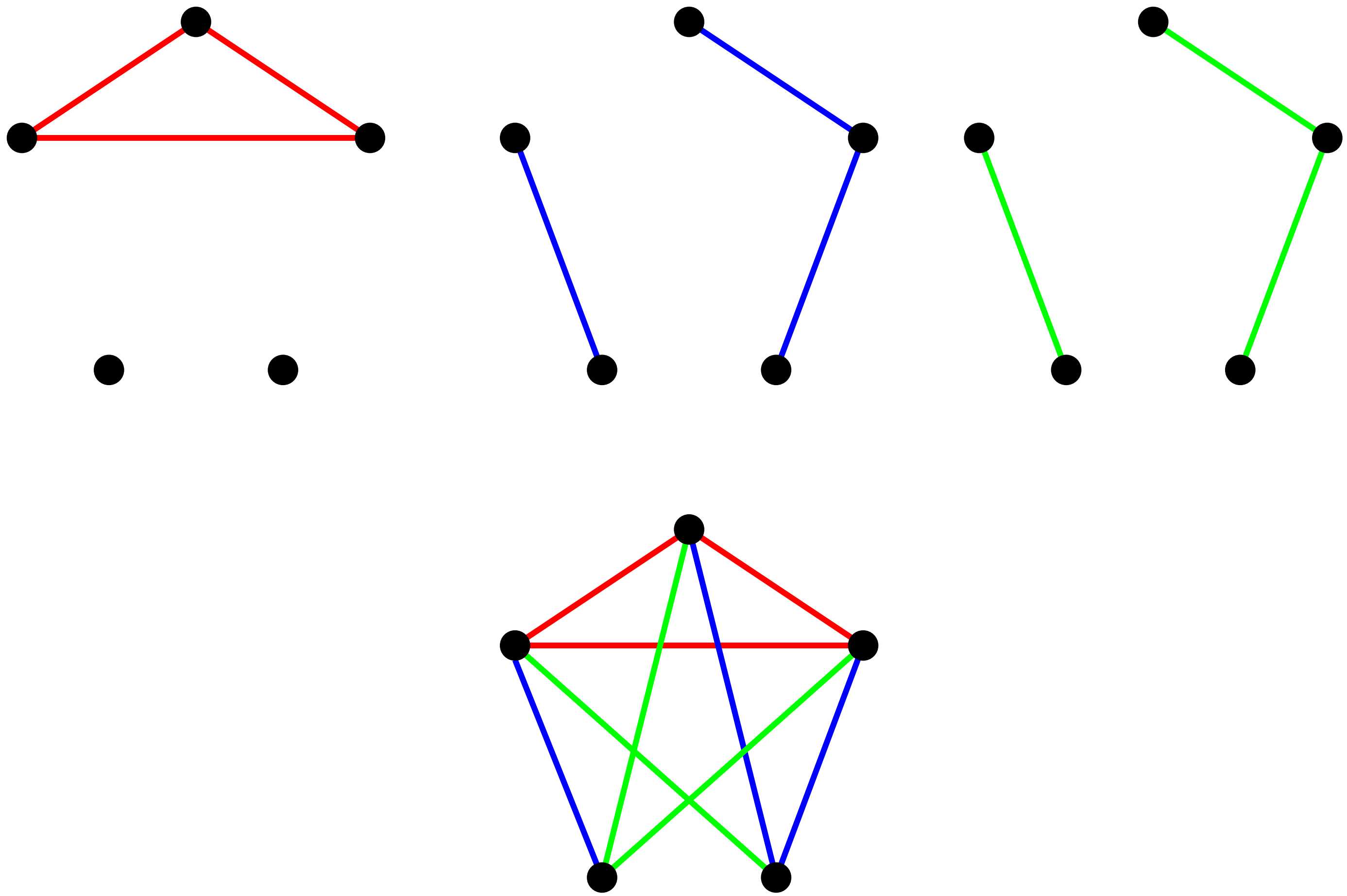
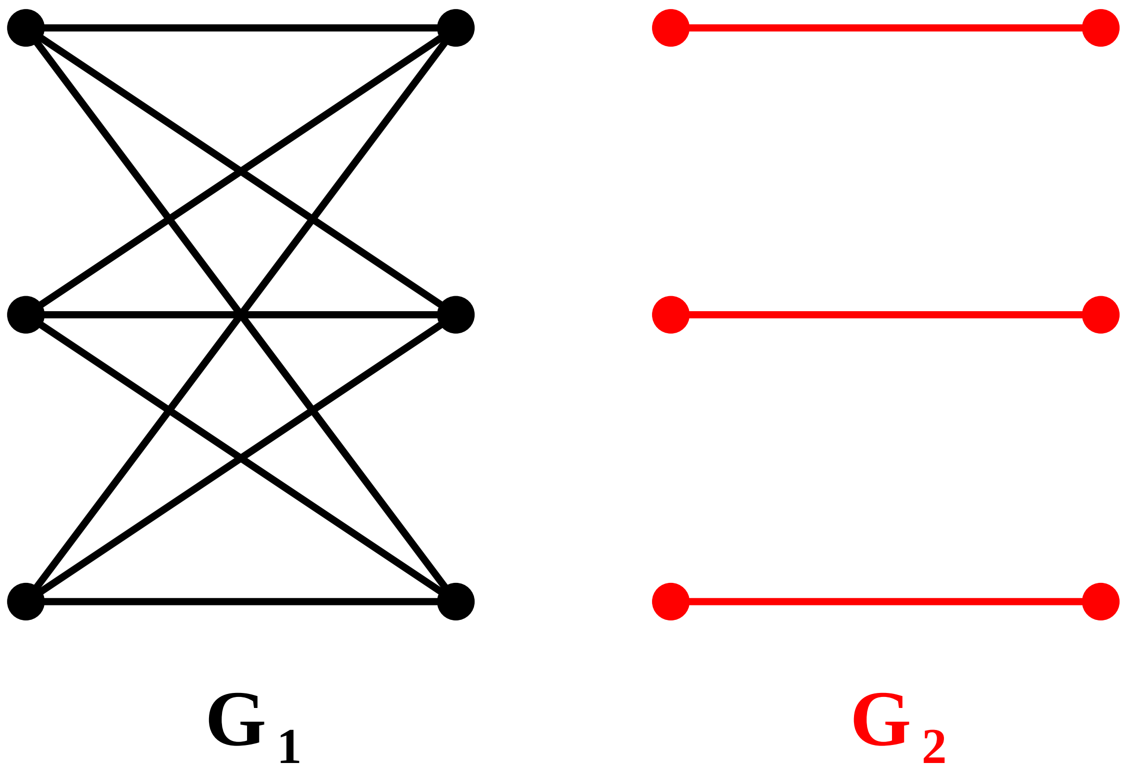
or, equivalently, the complement, $\overline{G}_2$, of G_2 contains G_1.

Example 1: Three graphs that pack



Example 2: Two graphs that do not pack



Examples of packing problems

• An n-vertex graph G has a Hamiltonian cycle $\Longleftrightarrow$ $\overline{G}$ packs with the n-cycle.

• A graph G is k-colorable $\Longleftrightarrow$ G packs with the complement of a complete k-partite graph.

Let $H(n,k)$ denote the n-vertex graph every of whose k components is either a $\lfloor \frac{n}{k} \rfloor$-clique or a $\lceil \frac{n}{k} \rceil$-clique.

• An n-vertex graph G has an equitable k-coloring $\Longleftrightarrow$ G packs with $H(n,k)$.

Some known theorems in the language of packing

Dirac's Theorem. Every n-vertex graph G with $\Delta(G) \leq n/2 - 1$ packs with the n-cycle.

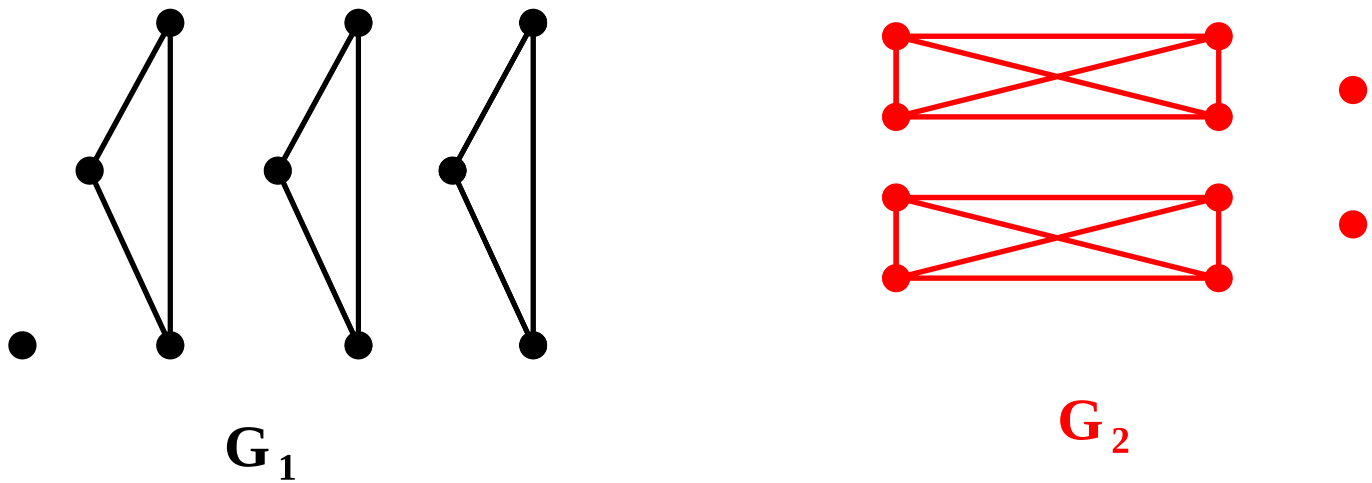
Ore's Theorem. Every n-vertex graph G with $d(x) + d(y) \leq n - 2$ for each edge xy packs with the n-cycle.

Brooks' Theorem. If $r \geq 3$, G is a connected graph with $\Delta(G) \leq r$, and G does not pack with the complement of any r-partite graph, then $G = K_{r+1}$.

Hajnal-Szemerédi Theorem. Every n-vertex graph G with $\Delta(G) \leq k - 1$ packs with $H(n, k)$.

Conjecture 1. Bollobás and Eldridge, Catlin *If* G_1 *and* G_2 *are* n*-vertex graphs and* $(\Delta(G_1)+1)(\Delta(G_2)+1) \leq n+1$*, then* G_1 *and* G_2 *pack.*

This is the BEC-conjecture. The Hajnal-Szemerédi Theorem is a partial case of it.



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$$\delta(G) + \Delta(G) \leq \theta(G) \leq 2\Delta(G).$$

$$\theta(G) = \Delta(L(G)) + 2.$$

We call $\theta(G)$ *Maximum total edge-degree* because it equals the maximum degree in the total graph $T(G)$ of a ”vertex” that was an edge in G.

An Ore-type conjecture

Conjecture 2. Kostochka and Yu *If* $\theta(G) \leq 2r$, *then* G *has an equitable* $(r+1)$*-coloring.*

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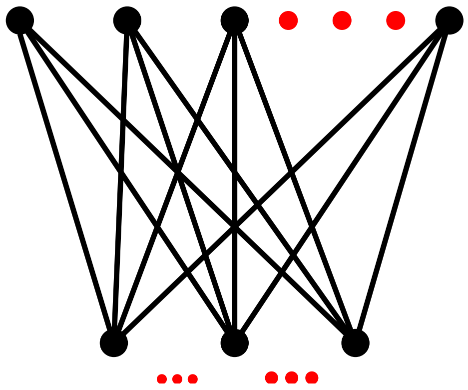
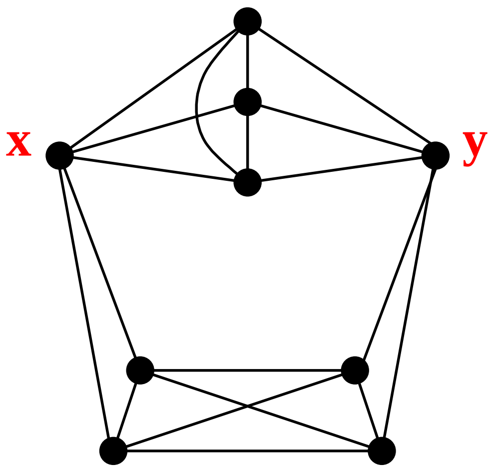
Theorem 1. Kierstead and Kostochka *If* $\theta(G) \leq 2r+1$*, then* G *has an equitable* $(r+1)$*-coloring.*

Theorem 1 is sharp



$K_{m,2r-m}$ (odd m)



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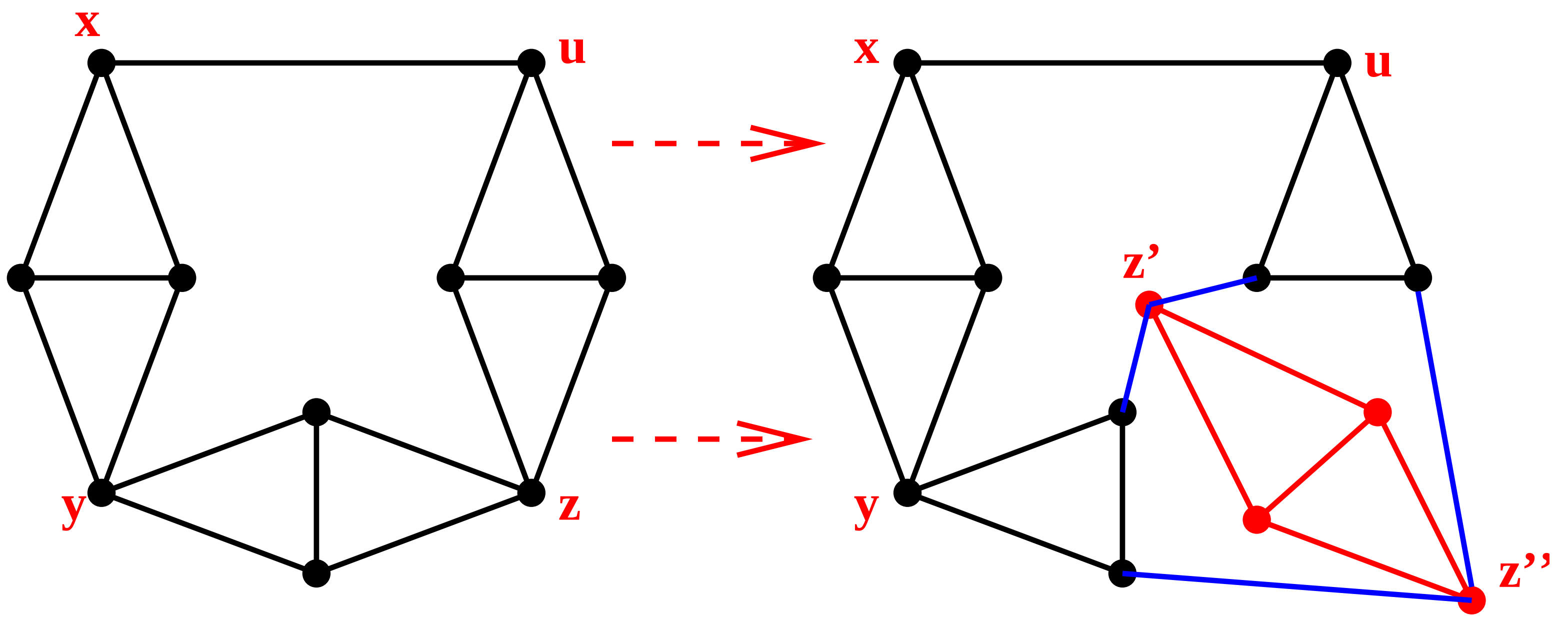

Theorem 2. [K-K] *If* $r \geq 6$, $\theta(G) \leq 2r + 1$ *and* G *does not contain* K_{r+1}, *then*

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Conjecture 3. [K-K] If $r \geq 3$ and a connected graph G with $\theta(G) \leq 2r$ differs from K_{r+1} and $K_{m,2r-m}$ for all odd m, then

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Theorem 3. [K-K] Conjecture 3 holds for $r = 3$.

For odd $r \geq 3$, the Chen-Lih-Wu Conjecture does not describe disconnected graphs with max.deg r that are not equitably r-colorable. For example, for an odd r, $K_{r,r} \cup K_{r,r}$ is equitably r-colorable, but $K_{r,r} \cup K_r$ is not.

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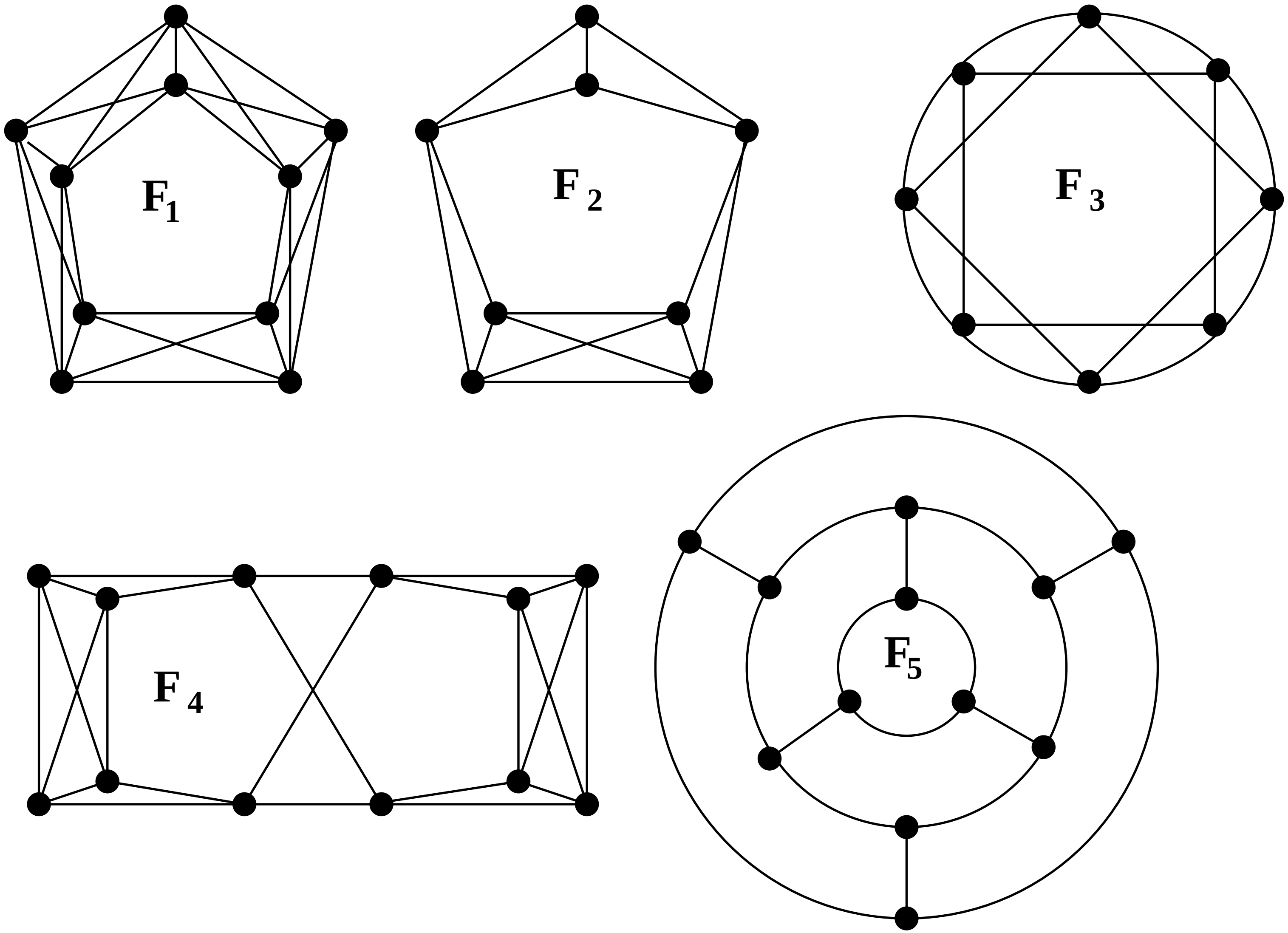
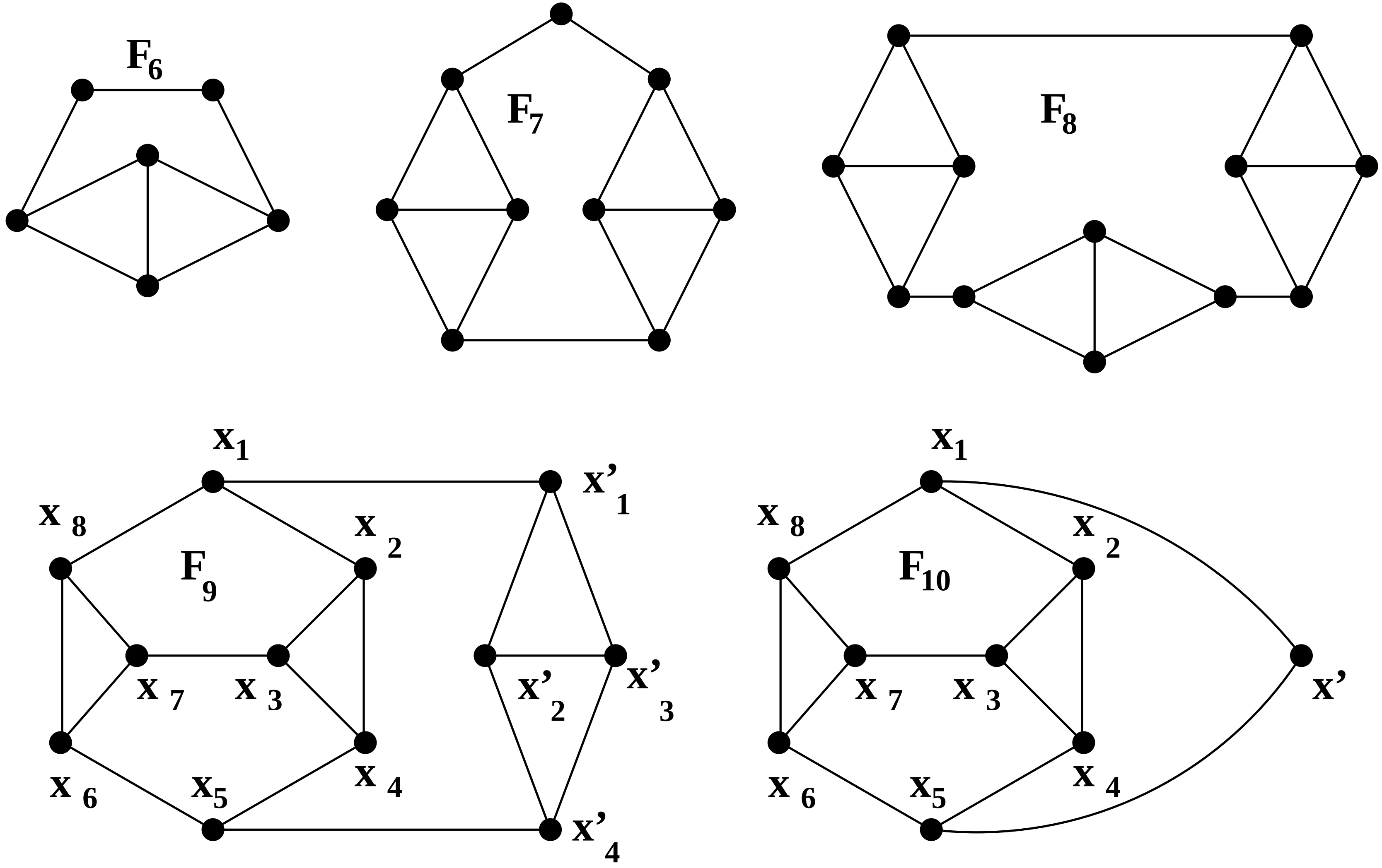
Clearly, a graph G can be r-equitable only for one r. Call G equitable if it is r-equitable for some r.

Basic equitable graphs



More basic equitable graphs



Theorem 4. [K-K] If $r \geq 3$ and a graph G with $\theta(G) \leq 2r$ and $|V(G)|$ divisible by r has an equitable r-coloring but has no nearly equitable r-coloring, then G is vertex-decomposable into r-basic graphs.

Conjecture 4. [K-K] If $r \geq 3$ is odd, then an r-colorable graph G with $\Delta(G) \leq r$ does not have an equitable r-coloring if and only if a spanning subgraph of G is the disjoint union of $K_{r,r}$ and basic r-equitable graphs.

Theorem 5. [K-K] For every odd $r \geq 3$, if the Chen-Lih-Wu Conjecture holds for graphs G with $\Delta(G) \leq r$, then Conjecture holds for graphs G with $\Delta(G) \leq r$.

Corollary 6. [K-K] Conjecture 4 holds for $r = 3$.

Conjecture 5. [K-K] If $r \geq 3$, then an r-colorable graph G with $\theta(G) \leq 2r$ does not have an equitable r-coloring if and only if a spanning subgraph of G is the disjoint union of $K_{m,2r-m}$ for some odd m and basic r-equitable graphs.

Theorem 7. [K-K] For every $r \geq 3$, if Conjecture 3 holds for graphs G with $\theta(G) \leq r$, then Conjecture 5 holds for graphs G with $\theta(G) \leq r$.

Corollary 8. [K-K] Conjecture 5 holds for $r = 3$.

Theorem 9. [K-K] The CLW-Conjecture holds for $r = 4$.