

# Many Disjoint Dense Subgraphs versus Large $k$ -connected Subgraphs in Large Graphs with Given Edge Density

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## Abstract

It is proved that every graph with edge density at least  $d + k$  and sufficiently many vertices contains a  $k$ -connected subgraph on at least  $t$  ( $t \geq k+1$ ) vertices, or  $s$  pairwise disjoint subgraphs with edge density at least  $d$ . By a classical result of Mader [11] this implies that every graph with edge density at least  $3k$  and sufficiently many vertices contains a  $k$ -connected subgraph with at least  $r$  vertices, or  $r$  pairwise disjoint  $k$ -connected subgraphs. Another classical result of Mader [10] states that for every  $n$  there is an  $l(n)$  such that every graph with edge density at least  $l(n)$  contains a minor isomorphic to  $K_n$ . Recently, it was proved in [1] that every  $(\frac{31}{2}a+1)$ -connected graph with sufficiently many vertices either has a topological minor isomorphic to  $K_{a,pq}$ , or it has a minor isomorphic to the disjoint union of  $p$  copies of  $K_{a,q}$ . Combining these results with the main result of the present note shows that every graph with edge density at least  $l(a) + (\frac{31}{2}a+1)$  and sufficiently many vertices has a topological minor isomorphic to  $K_{a,pa}$ , or a minor isomorphic to the disjoint union of  $p$  copies of  $K_a$ . This implies an affirmative answer to a question of Fon-der-Flaass.

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# 1 Introduction and Results

In this note all graphs are finite and do not have loops or multiple edges. For a graph  $G$ , let  $|G|$  and  $\|G\|$  denote the number its vertices and edges, respectively. The quotient  $\|G\|/|G|$  is called the *edge density* of  $G$ . A *separation* of a graph  $G$  is a pair  $(G_1, G_2)$  of subgraphs  $G_1, G_2$  of  $G$  such that  $G = G_1 \cup G_2$ ,  $E(G_1 \cap G_2) = \emptyset$ . The *order* of a separation  $(G_1, G_2)$  is  $|G_1 \cap G_2|$ . A separation  $(G_1, G_2)$  is *non-trivial* if  $V(G_1) \setminus V(G_2) \neq \emptyset$  and  $V(G_2) \setminus V(G_1) \neq \emptyset$ . We call a graph *k-separable* if it has a non-trivial separation of order at most  $k$ . Clearly, a graph is not  $(k - 1)$ -separable if and only if it is either  $k$ -connected or it is a complete graph with at most  $k$  vertices. Our main result is the following Theorem 1.1.

**Theorem 1.1** *There exist a function  $M : \mathbb{N}^4 \rightarrow \mathbb{N}$  such that every graph with edge density at least  $d + k$  and at least  $M(d, k, s, t)$  vertices contains a not  $(k - 1)$ -separable subgraph with at least  $t$  vertices, or  $s$  pairwise disjoint subgraphs with edge density at least  $d$ .*

Our first application of Theorem 1.1 deals with the existence of  $k$ -connected subgraphs in graphs with given edge density. Mader [11] (see also [4], p. 11) proved that every graph with edge density at least  $2k$  contains a  $k$ -connected subgraph. Mader's result together with Theorem 1.1 immediately implies the following Theorem 1.2. (Put  $d = 2k$ ,  $s = t = r$ , and  $N(k, r) = M(2k, k, r, r)$ .)

**Theorem 1.2** *There exist a function  $N : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$  such that for  $r \geq k + 1$  every graph with edge density at least  $3k$  and at least  $N(k, r)$  vertices contains a  $k$ -connected subgraph with at least  $r$  vertices, or  $r$  pairwise disjoint  $k$ -connected subgraphs.*

For a second application, we use Theorem 1.1 to prove a result on the existence of certain (topological) minors in graphs with given edge density. A graph  $H$  is a *topological minor* of a graph  $G$  if it can be obtained from a subgraph  $G'$  of  $G$  by replacing the edges of  $G'$  with paths between their ends, such that none of these paths has an inner vertex on another path or in  $G'$ . A graph  $H$  is a *minor* of another graph  $G$  if  $H$  can be obtained from a subgraph of  $G$  by contracting edges. Clearly, every topological minor of a graph is also a minor of this graph.

Mader [9, 10] proved that there exist functions  $h, l : \mathbb{N} \rightarrow \mathbb{R}$  such that every graph with edge density at least  $h(n)$  has a topological minor isomorphic to the complete graph  $K_n$ , and every graph with edge density at least  $l(n)$

has a minor isomorphic to the complete graph  $K_n$ . It was proved independently by Bollobás and Thomason [3], and by Komlós and Szemerédi [6] that  $h(n) = O(n^2)$ . Kostochka [8, 7] and Thomason [12] independently proved that  $l(n) = \Theta(n\sqrt{\log n})$ . Recently, Thomason [13] found the asymptotically best possible value of  $l(n)$ .

Applying Theorem 1.1 with  $d = h(n)$  (resp.  $d=l(n)$ ) shows that for  $t \geq k + 1$  every graph with edge density at least  $h(d) + k$  (resp.  $l(d) + k$ ) and at least  $M(h(d), k, s, t)$  (resp.  $M(l(d), k, s, t)$ ) vertices contains a  $k$ -connected subgraph with at least  $t$  vertices, or it has a topological minor (resp. a minor) isomorphic to the disjoint union of  $s$  copies of the complete graph  $K_n$ . We will combine this result with a very recent theorem of Böhme, Kawarabayashi, Maharry, and Mohar [1].

**Theorem 1.3 ([1])** *There exists a function  $L : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$  such that every  $(\frac{31}{2}a + 1)$ -connected graph with at least  $L(p, q)$  vertices either has a topological minor isomorphic to  $K_{a,pq}$ , or it has a minor isomorphic to the disjoint union of  $p$  copies of  $K_{a,q}$ .*

We let  $P(a, p, q) = M(l(a), (\frac{31}{2}a + 1), p, L(p, a))$  and obtain the following Theorem 1.4. (Note that  $K_a$  is a minor of  $K_{a,a}$ .)

**Theorem 1.4** *There is a function  $P : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$  such that every graph with edge density at least  $l(a) + (\frac{31}{2}a + 1)$  and at least  $P(a, p)$  vertices has a topological minor isomorphic to  $K_{a,pa}$ , or a minor isomorphic to the disjoint union of  $p$  copies of  $K_a$ .*

Since  $K_{a,pa}$  contains vertices of degree  $pa$ , Theorem 1.4 implies the following Corollary 1.5 which answers a question of D. Fon-der-Flaass [5].

**Corollary 1.5** *Every graph with edge density at least  $l(a) + (\frac{31}{2}a + 1)$ , maximum degree at most  $pa - 1$  and at least  $P(a, p)$  vertices has a minor isomorphic to the disjoint union of  $p$  copies of  $K_a$ .*

## 2 Proofs

We will prove the following variation of Theorem 1.1.

**Theorem 2.1** *For any positive integers  $d, k, s, t$  there is a number  $M = M(d, k, s, t)$  such that every graph on  $n$  vertices with at least  $(d+k-1)n + M$  edges contains either a not  $(k-1)$ -separable subgraph with at least  $t$  vertices or  $s$  pairwise disjoint subgraphs with edge density at least  $d$ .*

**Proof of Theorem 2.1**

Let  $M(d, k, s, t) = 2^{s-1} \binom{t}{2} + (2^{s-1} - 1)(d + k - 1)(k - 1)$ . If  $\|G\| \geq (d + k - 1)|G| + M(d, k, 1, t)$ , then the graph  $G$  itself has edge density greater than  $d$ , and so the statement of the theorem is trivially true for  $s = 1$ .

Suppose there is an  $s > 1$  such that the statement of the theorem is not true, and let  $G$  be a graph such that

- (a)  $\|G\| \geq (d + k - 1)|G| + 2^{s-1} \binom{t}{2} + (2^{s-1} - 1)(d + k - 1)(k - 1)$ ;
- (b)  $G$  contains neither a not  $(k - 1)$ -separable subgraph on at least  $t$  vertices nor  $s$  pairwise disjoint subgraphs with edge density at least  $d$ ;
- (c)  $s$  is minimum with respect to (a) and (b);
- (d)  $|G|$  is minimum with respect to (a), (b) and (c).

Since  $G$  has more than  $\binom{t}{2}$  edges,  $|G| > t$ . Hence it follows from (b) that  $G$  has a non-trivial separation  $(G_1, G_2)$  of order at most  $k - 1$ .

**Claim 1** *There is an  $i \in \{1, 2\}$  such that*

$$\|G_i\| \geq (d + k - 1)|G_i| + 2^{s-2} \binom{t}{2} + (2^{s-2} - 1)(d + k - 1)(k - 1).$$

**Proof of Claim 1.** Suppose that for  $i \in \{1, 2\}$ ,

$$\|G_i\| < (d + k - 1)|G_i| + 2^{s-2} \binom{t}{2} + (2^{s-2} - 1)(d + k - 1)(k - 1).$$

Then, since  $|G_1| + |G_2| \leq |G| + (k - 1)$ , we have

$$\begin{aligned} \|G\| &= \|G_1\| + \|G_2\| < \\ &(d + k - 1)(|G_1| + |G_2|) + 2^{s-1} \binom{t}{2} + (2^{s-1} - 2)(d + k - 1)(k - 1) \leq \\ &(d + k - 1)|G| + 2^{s-1} \binom{t}{2} + (2^{s-1} - 1)(d + k - 1)(k - 1) - \text{a contradiction.} \end{aligned}$$

□

Assume w.l.o.g. that

$$\|G_2\| \geq (d + k - 1)|G_2| + 2^{s-2} \binom{t}{2} + (2^{s-2} - 1)(d + k - 1)(k - 1),$$

and let  $G_1^- = G_1 - V(G_2)$ .

**Claim 2**  $\|G_1^-\| \geq d|G_1^-|$ .

**Proof of Claim 2.** Suppose that  $\|G_1^-\| < d|G_1^-|$ . Then

$$\begin{aligned} \|(G_2)\| &= \|G\| - \|G_1\| \geq \|G\| - \|G_1^-\| - (k-1)|G_1^-| > \\ (d+k-1)|G| - d|G_1^-| - (k-1)|G_1^-| + 2^{s-1} \binom{t}{2} + (2^{s-1}-1)(d+k-1)(k-1) = \\ (d+k-1)|G_2| + 2^{s-1} \binom{t}{2} + (2^{s-1}-1)(d+k-1)(k-1). \end{aligned}$$

Since  $|G_2| < |G|$ , this contradicts (d). □

Since  $G_2$  is a subgraph of  $G$  it follows from (b) that  $G_2$  does not contain a not  $(k-1)$ -separable subgraph on at least  $t$  vertices. This implies by (c) and Claim 1 that  $G_2$  contains  $s-1$  pairwise disjoint subgraphs with edge density at least  $d$ . Claim 2 implies that these subgraphs together with  $G_1^-$  form a family of  $s$  pairwise disjoint subgraphs of  $G$  with edge density at least  $d$ . This contradicts (b), and the theorem is proved. □

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