For a digraph $G = (V, E)$ and $v \in V$, denote by $E^+(v)$ the set of edges leaving v and by $E^-(v)$ the set of edges entering v.

A *network* $G = \{V, E, s, t, \mathbf{b} = \{b(e)\}_{e\in E}\}$ is a directed graph (V, E) with a *source vertex* s, a *sink vertex* t, and non-negative *capacities* $\{b(e)\}_{e\in E}$ of edges. A function $f : E \to \mathbf{R}$ is called *a flow* if for every vertex $v \in V - s - t$,

$$div_f(v) = \sum_{e\in E^+(v)} f(e) - \sum_{e\in E^-(v)} f(e) = 0. \tag{1}$$

If $0 \le f(e) \le b(e)$ for every $e \in E$, then the flow is called *feasible* (for G).

Consider $\sum_{v\in V} div_f(v)$. If f is a flow, then by (1) this sum equals $div_f(s)+div_f(t)$. On the other hand, every edge $e = uv$ contributes $f(e)$ to $div_f(u)$ and $-f(e)$ to $div_f(v)$. Therefore, $div_f(s) + div_f(t) = 0$. The value $M(f) = div_f(s) = -div_f(t)$ is called *the value of* f. A flow with value zero is called *circulation*.

Clearly, for any flows f and g and any reals α and β, $M(\alpha f + \beta g) = \alpha M(f) + \beta M(g)$.

A flow f is *positive* if $f(e) \ge 0$ for every $e \in E$ and there exists $e_0 \in E$ such that $f(e_0) > 0$. We say that a flow f is a flow *along a(n oriented) cycle* (or *along a(n oriented) s,t-path*) if f is non-zero only on edges of this cycle (s,t-path).

Lemma 1 *Every positive circulation f in a network G can be represented as the sum of at most $|E(G)| - 1$ positive flows along cycles.*

PROOF. Let the pair (G, f) be a counterexample with the minimum possible number of edges in G. If $f(e_0) = 0$ for some $e_0 \in E$, then consider $G_0 = G - e_0$ and $f\mid_{G_0}$. By the minimality of G we are done.

Thus we may assume that $f(e) > 0$ for every $e \in E$. Consider an arbitrary $e_1 = v_0v_1 \in E$. Since f is a circulation, there is an edge $e_2 = v_1v_2$ leaving v_1. Similarly, there exists an edge $e_3 = v_2v_3$ leaving v_2, and so on. Let k be the minimum positive integer such that $v_k \in \{v_0, \dots, v_{k-1}\}$. For definiteness, let $v_k = v_m$. Then $C = v_mv_{m+1}\dots v_{k-1}v_k$ is a cycle in G. Let $\rho = \min\{f(e) \mid e \in E(C)\}$, and $\varphi_C(\rho)$ be the flow along C of size ρ. Consider $f_1 = f - \varphi_C(\rho)$. If $f_1 \equiv 0$, we are done. Otherwise, f_1 is a positive flow and there exists $e_1 \in E(C)$ with $f_1(e_1) = 0$. Due to the minimality of G, the flow $f_1\mid_{G-e_1}$ can be represented as the sum of at most $|E(G - e_1)| - 1 = |E(G)| - 2$ positive flows along cycles. Adding $\varphi_C(\rho)$ to this sum, we find a representation for f, a contradiction. This proves the lemma.

Theorem 1 *Every positive flow f in a network G can be represented as the sum of at most $|E(G)|$ positive flows along cycles, s,t-paths and t,s-paths.*

PROOF. CASE 1. $M(f) > 0$. Let G_0 be obtained from G by adding new edge $e_0 = ts$ and let f_0 differ from f only in that $f_0(e_0) = M(f)$. Then f_0 is a circulation in G_0. By Lemma 1, f_0 is the sum of at most $|E(G_0)| = |E(G)|$ positive flows along cycles. These flows correspond in G to flows along cycles and paths from s to t.

CASE 2. $M(f) < 0$. Let G_0 be obtained from G by adding new edge $e_0 = st$ and let f_0 differ from f only in that $f_0(e_0) = -M(f)$. Then the argument is the same as in Case 1. This proves the theorem.