1 Counting mappings

For every real x and positive integer k, let $[x]_k$ denote *the falling factorial*

$$x(x-1)(x-2)\dots(x-k+1)$$

and

$$\binom{x}{k} = \frac{[x]_k}{k!}, \quad \binom{k}{0} = 1.$$

In the sequel, $X = \{x_1, \dots, x_m\}, Y = \{y_1, \dots, y_n\}$. If we write $\widetilde{X}$ (resp., $\widetilde{Y}$), that means that we cannot distinguish the elements of X (resp., Y).

Proposition 1 *The total number of all mappings $f : X \to Y$ is n^m and the number of injective mappings $f : X \to Y$ is $[n]_m$.*

Proposition 2 *The number of bijective mappings $f : X \to Y$ is $n!$ when $n = m$ and 0 otherwise.*

Proposition 3 *The number of injective mappings $f : \widetilde{X} \to Y$ is $\binom{n}{m}$.*

PROOF. Associate with every injective mapping $f : X \to Y$ the image, $f(X)$, of X. By Proposition 1, the number of injective mappings $f : X \to Y$ is $[n]_m$. By Proposition 2, every subset Z of Y with $|Z| = m$ is the image of X for $m!$ injective mappings, and these mappings are indistinguishable in the model $f : \widetilde{X} \to Y$. Thus, the number in question is $[n]_m/m! = \binom{n}{m}$. ∇

Proposition 4 *The number of surjective mappings $f : \widetilde{X} \to Y$ is $\binom{m-1}{n-1}$.*
The total number of all mappings $f : \widetilde{X} \to Y$ is $\binom{n+m-1}{n-1}$.

PROOF. Assign a set of $n-1$ vertical strips between m points points on a line to every surjective mapping $f : \widetilde{X} \to Y$ as follows. If $|f^{-1}(y_i)| = a_i$, then put the i-th strip between the points with the numbers $a_1 + \ldots + a_i$ and $a_1 + \ldots + a_i + 1$. Observe that this is a bijection between the set of surjective mappings $f : \widetilde{X} \to Y$ and the set of arrangements of $n-1$ strips between m points on a line such that each interval between two consecutive points can contain at most one strip. But the number of such arrangements is $\binom{m-1}{n-1}$. ∇

Similarly, we have a bijection between the set of all mappings $f : \widetilde{X} \to Y$ and the set of all possible arrangements of $n-1$ strips between m points on a line. And the number of such arrangements is $\binom{n+m-1}{n-1}$.

The number $S(m, k)$ of the partitions of an m-element set into k non-empty subsets is a *Stirling number of the second kind.*

Proposition 5 *(1) The number of surjective mappings $f : X \to \widetilde{Y}$ is $S(m, n)$.*
(2) The number of surjective mappings $f : X \to Y$ is $n!S(m, n)$.
(3) The number of all mappings $f : X \to \widetilde{Y}$ is $\sum_{k=1}^{n} S(m, k)$.

PROOF. (1) is easy.

Every surjective mapping $f : X \to Y$ can be constructed in two steps. On the first one we partition X into n parts that will be mapped into the same element of Y (by the definiton, there is $S(m,n)$ ways to do this), and on the second step we choose the image for every of these n parts ($n!$ ways). This proves (2).

Now, (3) follows from (1).

Proposition 6 *(1)* $S(m,m) = 1$ *for* $m \geq 0$.
(2) $S(m,0) = 0$ *for* $m > 0$.
(3)$S(m,k) = S(m-1,k-1) + kS(m-1,k)$ *for* $0 < k < m$.

PROOF. (1) and (2) are evident. To see (3), observe that the set R of all partitions of $X = \{x_1, \ldots, x_m\}$ into n non-empty parts is the union of $R_1 = \{\pi \in R \mid \{x_m\}$ is a part of the partition $\}$ and $R_2 = R \backslash R_1$.

Clearly, $|R_1| = S(m-1,k-1)$. Assign every $\pi \in R_2$ the partition π' of $X \backslash \{x_m\}$, obtained from π by deleting x_m. Notice that every partition π' of $X \backslash \{x_m\}$ into k parts is assigned to k different partitions of X. This shows that $|R_2| = kS(m-1,k)$. ∇

This proposition lets us count $S(m,k)$ recursively:

$m \setminus k$	0	1	2	3	4	5	6	7
1	0	1						
2	0	1	1					
3	0	1	3	1				
4	0	1	7	6	1			
5	0	1	15	25	10	1		
6	0	1	31	90	65	15	1	
7	0	1	63	301	350	140	21	1

Proposition 7 *For every* $k \geq 2$

$$S(m,k) = \sum_{i=k-1}^{m-1} \binom{m-1}{i} S(i,k-1).$$

PROOF. Let $(X_1, \ldots, X_k)$ be an arbitrary partition of X into k non-empty parts and $x_m \in X_k$. There are $\binom{m-1}{r-1}$ ways to choose X_k with $|X_k| = r$ and $x_m \in X_k$. Now there are $S(m-r,k-1)$ ways to partition the rest into $k-1$ non-empty parts. Hence,

$$S(m,k) = \sum_{r=1}^{m-k+1} \binom{m-1}{r-1} S(m-r,k-1) = \sum_{i=k-1}^{m-1} \binom{m-1}{i} S(i,k-1). \qquad \nabla$$

Theorem 8 *For every integer* $m \geq 0$,

$$x^m = \sum_{k=1}^{m} S(m,k)[x]_k.$$

PROOF. Let n be a positive integer. By Proposition For every $A \subseteq Y$, $A \neq \emptyset$, let $\mathcal{X}_A$ be the set of mappings $f : X \to Y$ such that $f(X) = A$. If $|A| = k$, then $|\mathcal{X}_A| = k!S(m,k)$. Hence by Proposition 5 (3),

$$n^m = \sum_{k=1}^{m} \binom{n}{k} S(m,k)k! = \sum_{k=1}^{m} S(m,k)[n]_k.$$

Thus the values of the polynomials x^m and $\sum_{k=1}^{m} S(m,k)[x]_k$ coincide for $x = 1, 2, 3, \ldots$. It follows that this is the same polynomial. ∇

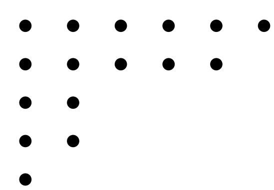
Now, consider mappings $\widetilde{X}$ into $\widetilde{Y}$. In order to distinguish two such mappings f_1 and f_2, we need the sets $\{|f_1^{-1}(y_1)|, |f_1^{-1}(y_2)|, \ldots, |f_1^{-1}(y_n)|\}$ and $\{|f_2^{-1}(y_1)|, |f_2^{-1}(y_2)|, \ldots, |f_2^{-1}(y_n)|\}$ to be different. Thus, the number $\widetilde{R}(m,n)$ of all mappings of $\widetilde{X}$ into $\widetilde{Y}$ is the number of partitions of the number m into the sum of n non-negative integers.

Let $P(m,k)$ denote the number of partitions of the number m into the sum of k positive integers, and $P(m)$ denote the number of partitions of the number m into the sum of positive integers. Clearly,

$$P(m) = \sum_{k=1}^{m} P(m,k) \qquad \forall m > 0,$$

$$\widetilde{R}(m,n) = \sum_{k=1}^{n} P(m,k).$$

Helpful tools in studying $P(m,k)$ are Ferrer's diagrams. A *Ferrer's diagram* for the partition $m = a_1 + a_2 + \ldots + a_k$ (assuming $a_1 \geq a_2 \geq \ldots \geq a_k$) consists of k rows, and the i-th row contains a_i points in columns $1, \ldots, a_i$. The example below shows the Ferrer's diagram for the partition $16 = 6 + 5 + 2 + 2 + 1$.



Ferrer's diagrams imply the following proposition.

Proposition 9 *(1) The number of partitions of m into the sum of exactly k positive integers equals the number of partitions of m into the sum of positive integers the maximum of which is k.*
(2) $P(m,n) = P(m-n,1) + P(m-n,2) + \ldots + P(m-n,n)$.

Now we prove a deeper fact

Proposition 10 *The number of partitions of m into the sum of distinct positive integers equals the number of partitions of m into the sum of odd positive integers.*

PROOF. Let $\mathcal{Q}$ be the set of partitions of m into the sum of distinct positive integers and $\mathcal{P}$ be the set of partitions of m into the sum of odd positive integers. Construct the mapping $F : \mathcal{Q} \to \mathcal{P}$ as follows. Let $m = a_1 + a_2 + \ldots + a_k$ and $a_1 > a_2 > \ldots > a_k$. Write every a_i in the form $a_i = 2^{c_i} b_i$, where b_i is odd. There is only one way to do so. Map the partition $m = a_1 + a_2 + \ldots + a_k$ to the partition

$$m = \underbrace{b_1 + b_1 + \ldots + b_1}_{2^{c_1} \text{ times}} + \underbrace{b_2 + b_2 + \ldots + b_2}_{2^{c_2} \text{ times}} + \ldots + \underbrace{b_k + b_k + \ldots + b_k}_{2^{c_k} \text{ times}}.$$

This is our mapping F.

Now we construct the mapping $G : \mathcal{P} \to \mathcal{Q}$ as follows. Let a partition

$$m = \underbrace{b_1 + b_1 + \ldots + b_1}_{d_1 \text{ times}} + \underbrace{b_2 + b_2 + \ldots + b_2}_{d_2 \text{ times}} + \ldots + \underbrace{b_s + b_s + \ldots + b_s}_{d_s \text{ times}} \qquad (1)$$

of m into the sum of odd positive integers be given, and let there be exactly s distinct summands. Every of d_i-s can be written in the form

$$d_i = 2^{e_{i,1}} + 2^{e_{i,2}} + \ldots + 2^{e_{i,t(i)}}, \quad e_{i,1} > e_{i,2} > \ldots > e_{i,t(i)}. \qquad (2)$$

Now the partition

$$m = \sum_{i=1}^{s} (2^{e_{i,1}} b_i + 2^{e_{i,2}} b_i + \ldots + 2^{e_{i,t(i)}} b_i) \qquad (3)$$

consists of distinct summands. Indeed, for the same i, the summands are distinct since $e_{i,j}$ are distinct, and for distinct i, the maximum odd divisors are distinct. Let G map every partition of the kind (1) to the partition of the kind (3).

Observe that if we apply G to a partition of the kind (1) and then apply F to the obtained partition, then we get the initial partition. Since the form (2) is uniquely determined, F maps different partitions of m into the sum of distinct positive integers into different partitions of m into the sum of odd positive integers. It follows that F and G are bijections and $|\mathcal{P}| = |\mathcal{Q}|$. ∇

Part of the obtained results is summarized in the following table.

	All mappings	Injective mappings	Surjective mappings	Bijective mappings
$X \to Y$	n^m	$[n]_m$	$n!S(m,n)$	$n!$
$\widetilde{X} \to Y$	$\binom{n+m-1}{m}$	$\binom{n}{m}$	$\binom{m-1}{n-1}$	1
$X \to \widetilde{Y}$	$\sum_{k=1}^{n} S(m,k)$	0, if $m > n$ 1, if $m \leq n$	$S(m,n)$	1
$\widetilde{X} \to \widetilde{Y}$	$\sum_{k=1}^{n} P(m,k)$	0, if $m > n$ 1, if $m \leq n$	$P(m,n)$	1

2 Catalan's numbers

The *Catalan's numbers* $\{c_n\}_{n=0}^{\infty}$ are defined as follows: $c_0 = 1$ and for every $n > 0$,

$$c_n := c_0c_{n-1} + c_1c_{n-2} + \ldots + c_{n-1}c_0. \tag{4}$$

The sequence begins: $1, 1, 2, 5, 14, 42, 132, 429, \ldots$

Example 1. A *planar rooted tree* is a tree with a special vertex (called the *root*) where the sons of every vertex are linearly ordered. To every planar rooted tree T with the root r, we can correspond the pair (T_1, T_2) obtained from T by deleting the edge (r, s) connecting r with the oldest son s. We assume that T_1 contains r, and T_2 contains s. It is not hard to check that this is a 1-1 correspondence. Hence, for the number t_n of planar rooted trees with n edges, we have $t_0 = 1$ and for every $n > 0$,

$$t_n := t_0t_{n-1} + t_1t_{n-2} + \ldots + t_{n-1}t_0.$$

In other words, $t_n = c_n$ for every n.

Example 2. Let a planar rooted tree T be placed with root up on the plane and so that the sons of every vertex are placed from left to right in their order. Then the walk around this tree starting from the edge rs and leaving the tree on the left hand all the time corresponds to a sequence of 1-s and -1-s as follows: when we come from a father to a son, we put 1, otherwise, we put -1. We will get a sequence of n ones and n minus-ones, where for every $1 \leq i \leq 2n$, the sum of the first i entries is non-negative. Clearly, every sequence with these properties corresponds to planar rooted tree with n edges, and this correspondence is 1-1. It follows that the number p_n of such sequences is also c_n.

To calculate c_n, write p_n in the form $p_n = w_n - v_n$, where w_n is the number of all sequences with n ones and n minus-ones, and v_n is the number of sequences $(a_1, \ldots, a_{2n})$ of n ones and n minus-ones such that

$$\exists i : \qquad a_1 + \ldots + a_i \leq -1. \tag{5}$$

Clearly, $w_n = \binom{2n}{n}$.

In order to evaluate v_n, we correspond to every sequence $A = \{a_j\}_{j=1}^{2n}$ in the family $\mathcal{V}_n$ of sequences of n ones and n minus-ones satisfying (5), the sequence $F(A)$ of $n+1$ ones and $n-1$ minus-ones as follows.

Let $i(A)$ be the smallest positive integer i, such that $a_1 + \ldots + a_i = -1$. Then we get $F(A)$ by changing in A the signs of the first $i(A)$ members. One can observe (please, try it) that so defined F is a 1-1 correspondence between $\mathcal{V}_n$ and the set $\mathcal{Z}_n$ of all sequences of $n+1$ ones and $n-1$ minus-ones. Consequently, $|\mathcal{V}_n| = |\mathcal{Z}_n| = \binom{2n}{n-1}$ and thus

$$c_n = p_n = \binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1}\binom{2n}{n}.$$

Now we will derive this formula again using the generating function $C(t)$ of these numbers which we will find by the definition (4):

$$C(t) = \sum_{n=0}^{\infty} c_nt^n = 1 + \sum_{n=1}^{\infty}(c_0c_{n-1} + \ldots + c_{n-1}c_0)t^n =$$

$$= 1 + t\sum_{n=0}^{\infty}(c_0c_n + \ldots + c_nc_0)t^n = 1 + t(C(t))^2.$$

Solving the quadratic with respect to $C(t)$, we obtain

$$C(t) = \frac{1 \pm \sqrt{1-4t}}{2t}.$$

Since we know that $C(0) = 1$, we must take the minus in the formula.

Let us calculate the n-th derivative of $\sqrt{1-4t}$:

$$\frac{\mathbf{d}^n}{\mathbf{dt}^n}\sqrt{1-4t} = \binom{1/2}{n} n!(1-4t)^{1/2-n}(-4)^n = 2^n(-1)(2n-3)!!(1-4t)^{1/2-n}.$$

Hence

$$1 - \sqrt{1-4t} = \sum_{n=1}^{\infty} \frac{2^n(2n-3)!!}{n!} t^n$$

and

$$C(t) = \sum_{n=0}^{\infty} \frac{2^n(2n-1)!!}{(n+1)!} t^n.$$

Observe that

$$\frac{2^n(2n-1)!!}{(n+1)!} = \frac{(2n)!}{n!(n+1)!} = \frac{1}{n+1}\binom{2n}{n}.$$