CONFLICT-FREE COLORINGS OF UNIFORM HYPERGRAPHS WITH FEW EDGES

A. KOSTOCHKA, M. KUMBHAT, AND T. ŁUCZAK

Abstract. A coloring of the vertices of a hypergraph $\mathcal{H}$ is called *conflict-free* if each edge e of $\mathcal{H}$ contains a vertex whose color does not get repeated in e. The smallest number of colors required for such a coloring is called the conflict-free chromatic number of $\mathcal{H}$, and is denoted by $\chi_{CF}(\mathcal{H})$. Pach and Tardos studied this parameter for graphs and hypergraphs. Among other things, they proved that for an $(2r-1)$-uniform hypergraph $\mathcal{H}$ with m edges, $\chi_{CF}(\mathcal{H})$ is of the order of $m^{1/r}\log m$. They also raised the question whether the same result holds for r-uniform hypergraphs. In this paper we show that this is not necessarily true. Furthermore, we provide lower and upper bounds on the minimum number of edges of an r-uniform simple hypergraph that is not conflict-free k-colorable.

1. Introduction

Let $\mathcal{H}$ be a hypergraph with vertex set $V(\mathcal{H})$ and edge set $E(\mathcal{H})$. A coloring $c: V(\mathcal{H}) \to \{1,2,3,...\}$ of $V(\mathcal{H})$ is a *proper coloring of* $\mathcal{H}$ if no edge of $\mathcal{H}$ is monochromatic. The minimum number of colors required for such a coloring is called the *chromatic number* of $\mathcal{H}$, and is denoted by $\chi(\mathcal{H})$. A *rainbow* coloring of $\mathcal{H}$ is a proper coloring of $\mathcal{H}$ such that for every edge e, the colors of all vertices of e are distinct. The minimum number of colors required for a rainbow coloring is called the *rainbow chromatic number* of $\mathcal{H}$, and is denoted by $\chi_R(\mathcal{H})$.

In connection with some frequency assignment problems for cellular networks, Even *et al.* [8] introduced (in a geometric setting) the following intermediate coloring. A proper coloring of $\mathcal{H}$ is *conflict-free* if for each edge e of $\mathcal{H}$, some color occurs on exactly one vertex of e. The minimum number of colors required for a conflict-free coloring is called the *conflict-free chromatic number* of $\mathcal{H}$, and is denoted by $\chi_{CF}(\mathcal{H})$. Because of applications and interesting behavior, this parameter attracted considerable attention (see, e.g. [2, 3, 4, 6, 8, 9, 13, 12]). In particular, Pach and Tardos [12] discussed the notion for general hypergraphs and proved several interesting results. Clearly, $\chi(\mathcal{H}) \leq \chi_{CF}(\mathcal{H}) \leq \chi_R(\mathcal{H})$ for every $\mathcal{H}$ with equalities when $\mathcal{H}$ is an ordinary graph. However, for general hypergraphs, the behavior of χ_{CF} may significantly differ from that of χ and of χ_R. For example, if we truncate an edge of a hypergraph, then χ cannot decrease, χ_R cannot increase, but χ_{CF} may increase, decrease, or stay

Date: April 17, 2010.

The work of the first author was supported in part by NSF grant DMS-0650784 and by grant 08-01-00673 of the Russian Foundation for Basic Research.

The third author wishes to thank the Foundation for Polish Science for its support.

Key words and phrases: Conflict-free, Hypergraph coloring, Simple hypergraph

2000 Mathematics Subject Classification: 05C15.

the same. Another example: if $\mathcal{H}$ is a 10^6-uniform hypergraph with 10 edges, then $\chi(\mathcal{H}) = 2$, $\chi_R(\mathcal{H}) = 10^6$, and $\chi_{CF}(\mathcal{H})$ can be either 2, 3, or 4.

A hypergraph is called *simple*, if any two distinct edges share at most one vertex. The *edge degree* of an edge e in a hypergraph $\mathcal{H}$ is the number of other edges intersecting e. The maximum edge degree $D(\mathcal{H})$ is the maximum of the edge degrees over all the edges of $\mathcal{H}$.

The chromatic number of hypergraphs is discussed in many papers. Some of them (e.g., [1, 5, 7, 11, 14, 15]) discuss the bounds on the chromatic number of uniform hypergraph in terms of their size or maximum (edge) degree. Pach and Tardos [12] analyzed the conflict-free colorings for graphs and hypergraphs. They proved that $\chi_{CF}(\mathcal{H}) \leq 1/2+\sqrt{2m+1/4}$ for every hypergraph with m edges, and that this bound is tight. They also showed the following result.

Theorem 1 ([12])**.** *Let $\mathcal{H}$ be a hypergraph with m edges such that the size of every edge is at least $2r-1$. Then $\chi_{CF}(\mathcal{H}) \leq C_r m^{1/r} \log m$, where C_r is a positive constant depending only on r.*

In fact, they proved the stronger result that the same bound holds for hypergraphs $\mathcal{H}$ in which the size of every edge is at least $2r-1$ and $D(\mathcal{H}) \leq m$. They also posed the question whether the same upper bound holds also when every edge has size at least r and intersects at most m others. In this paper we show that this is not necessarily true.

The goal of the paper is to give reasonable upper bounds on $\chi_{CF}(\mathcal{H})$ for r-uniform hypergraphs $\mathcal{H}$ with given number of edges or maximum edge degree. It will turn out that for a given m, the nature of the bounds for r-uniform hypergraphs with m edges significantly depends on whether r is small or large with respect to m. We also derive similar bound for simple r-uniform hypergraphs. It turns out that for positive integers r, k with $r \leq k/8$, both upper and lower bounds on the minimum number of edges in an r-uniform simple hypergraph that have no conflict-free colorings with k colors are roughly squares of the corresponding bounds for hypergraphs without the restriction of being simple.

For a warm-up, in Section 2 we find how few edges may r-uniform hypergraphs with χ_{CF} equal to 3 or 4 have. In particular, for arbitrarily large even r, there is an r-uniform hypergraph $\mathcal{H}$ with just 7 edges and $\chi_{CF}(\mathcal{H}) = 4$. In Section 3, we find upper bounds on $\chi_{CF}(\mathcal{H})$ in terms of the size/maximum edge degree of $\mathcal{H}$ and present some constructions showing that our bounds are reasonable. In Section 4, we do the same for simple r-uniform hypergraphs.

2. Conflict-free coloring of hypergraphs with very few edges

We define the *s-blow up* of a graph G to be the hypergraph formed by replacing every vertex v of G with s copies of v denoted by the set B_v, called a *blob*, and if uv is an edge in G, then $B_u \cup B_v$ is an edge in the blow-up.

Observation 2. *For a hypergraph $\mathcal{H}$, if either the degree of every vertex of H is at most 1, or if there is a vertex common to every edge of H, then $\chi_{CF}(\mathcal{H}) = 2$.*

Observation 3. *Let $r \geq 2$. If $\mathcal{H}$ is an r-uniform hypergraph which is not conflict-free 2-colorable, then it has at least 3 edges and the only such graph with 3 edges is the $\frac{r}{2}$-blow up of K_3.*

Proof. By Observation 2, every hypergraph with 2 edges is conflict-free 2-colorable. Moreover, a blow-up of K_3 is not. Now assume that $\mathcal{H}$ is an r-uniform hypergraph with 3 edges e_1, e_2, e_3 which is not conflict-free 2-colorable. If every vertex has degree at most 1 or if there is a vertex of degree 3, then by Observation 2 it is conflict-free 2-colorable. So assume that the maximum degree is 2. Without loss of generality assume that $v \in e_1 \cap e_2$. If there exists $u \in e_3 - e_1 - e_2$, then we color v and u with color 1 and the rest of the vertices with color 2. This would give a conflict-free 2-coloring of $\mathcal{H}$, a contradiction. Hence $e_3 \subseteq \{e_1 - e_2\} \cup \{e_2 - e_1\}$. Since $\mathcal{H}$ is r-uniform, we have that $e_3 \nsubseteq e_1$ and $e_3 \nsubseteq e_2$. Thus, $e_1 \cap e_3 \neq \emptyset$ and the above argument holds if v is replaced by a vertex $w \in e_3$. Consequently, $e_1 \subseteq \{e_2 - e_3\} \cup \{e_3 - e_2\}$ and, in a similar way, $e_2 \subseteq \{e_1 - e_3\} \cup \{e_3 - e_1\}$. Moreover, since $\mathcal{H}$ is r-uniform, it must be the $\frac{r}{2}$- blow up of K_3. In particular, r is even. □

Lemma 4. *Let $r \geq 3$. If $\mathcal{H}$ is an r-uniform hypergraph with at most 6 edges, then it is always conflict-free 3-colorable. Moreover, if $r \geq 4$ and divisible by 4, then there exists an r-uniform hypergraph with 7 edges which is not conflict-free 3-colorable.*

Proof. We first show that if $\mathcal{H}$ has at most 6 edges then it can always be conflict-free 3-colored.

Case 1. $\Delta(H) \geq 4$.

Let v be a vertex of degree at least 4. We color v with color 1. By Observation 3 w there is a conflict-free coloring of the edges not containing v with colors 2 and 3. We then color the remaining vertices with color 2. This gives a conflict-free 3-coloring of $\mathcal{H}$.

Case 2. $\Delta(H) \leq 2$.

Since for any hypergraph $\mathcal{G}$, $\chi_{CF}(\mathcal{G}) \leq \Delta(\mathcal{G})+1$ [12], we can conflict-free 3-color $\mathcal{H}$.

Case 3. $\Delta(H) = 3$.

Let v be a vertex of degree 3 belonging to the edges e_1, e_2 and e_3. If the other three edges $\{e_4, e_5, e_6\}$ are conflict-free 2-colorable, then we color them conflict-free with colors 2 and 3, color v with color 1 and arbitrarily color the remaining vertices with colors 2 and 3. This gives a conflict-free 3-coloring of $\mathcal{H}$. If not, then by Observation 3, $\{e_4, e_5, e_6\}$ form the $\frac{r}{2}$-blow up of K_3. We may assume that $e_4 \cup e_5 \cup e_6 = B_4 \cup B_5 \cup B_6$, where B_4, B_5, B_6 are the blobs $e_5 \cap e_6$, $e_4 \cap e_6$, $e_4 \cap e_5$, respectively. Now, suppose that there is a vertex $u \in \{e_4 \cup e_5 \cup e_6\} - \{e_1 \cup e_2 \cup e_3\}$. Without loss of generality assume that $u \in B_6$. Let w be a vertex in B_5. We now color v and u with color 1, w with color 2 and the rest of the vertices with color 3. This gives a conflict-free 3-coloring of $\mathcal{H}$. Hence $\{e_4 \cup e_5 \cup e_6\} \subseteq \{e_1 \cup e_2 \cup e_3\}$. Thus every vertex in $\{e_4 \cup e_5 \cup e_6\}$ has degree 3.

The above arguments hold for any vertex u in $\{e_4 \cup e_5 \cup e_6\}$ by replacing v with u and e_1, e_2, e_3 with the corresponding three edges u belongs to. Hence by symmetry, the degree of every vertex of $\mathcal{H}$ is 3. Moreover, since $\mathcal{H}$ is r-uniform, $\mathcal{H}$ must be the $\frac{r}{2}$-blow up of K_4. A blow-up of K_4 can be conflict-free 3-colored as follows. In the first blob we color a vertex with color 1 and another with color 2 and the rest 3. In the second blob we color one vertex with 2 and the rest 3. In the third blob we color one vertex with 1 and the rest 3 and in the fourth blob we color everything with color 3.

Now to show that there exists a hypergraph with 7 edges which is not 3-conflict-free colorable, we consider the $\frac{r}{4}$-blow up of the Fano plane and take the complement of every edge. Thus, the hypergraph has seven blobs $B_1, B_2, ...B_7$ and the following edges: $e_1 = B_1 \cup B_2 \cup B_6 \cup B_7$, $e_2 = B_2 \cup B_3 \cup B_4 \cup B_7$, $e_3 = B_4 \cup B_5 \cup B_6 \cup B_7$, $e_4 = B_1 \cup B_2 \cup B_4 \cup B_5$, $e_5 = B_1 \cup B_3 \cup B_4 \cup B_6$, $e_6 = B_2 \cup B_3 \cup B_5 \cup B_6$, and $e_7 = B_1 \cup B_3 \cup B_5 \cup B_7$. Suppose that it is conflict-free 3-colorable with colors $1, 2, 3$.

Claim 1: No color can appear in exactly one blob.
Proof: Let us assume that one color, say 1, appears in exactly one blob, say B_1 in the coloring. Consider the three edges e_2, e_3, e_6 not containing B_1. They must be conflict-free 2-colorable with colors $2, 3$. But they form the $\frac{r}{2}$-blow up of K_3 which is not conflict-free 2-colorable, a contradiction.

Claim 2: No color can appear in exactly two blobs.
Proof: Suppose that color 1 appears in exactly two blobs in the coloring. Let B_1, B_2 be the blobs containing vertices of color 1. Consider the two edges e_1, e_4 containing both B_1 and B_2 and the edge e_3 containing neither B_1 nor B_2. These three edges form the $\frac{r}{2}$-blow up of K_3 with at least two vertices of color 1 present in a single blob. All other vertices gets color 2 or 3. With these restrictions there exists no conflict-free 3-coloring of the blow up of K_3.

Hence by the above claims, every color appears in at least three blobs.

Since this hypergraph with seven edges is conflict-free 3-colorable, there must be a color which is unique for at least three edges. Assume that this color is 1.

We first claim that a vertex with color 1, cannot be unique for more than one edge. If not, then without loss of generality, assume that a vertex with color 1 belonging to B_1 is unique for edges e_4 and e_5. Hence the blobs B_2, B_3, B_4, B_5, B_6 do not have any vertices of color 1. Hence color 1 appears only in at most two blobs. By Claims 1 and 2, it cannot be conflict-free 3-colored. This proves the claim.

Assume that a vertex of color 1 in B_1 is unique for e_1. So the blobs B_2, B_6, B_7 do not have vertices of color 1. Again without loss of generality assume that a vertex of color 1 in B_3 is unique for the edge e_2. So the blob B_4 do not have any vertex of color 1. Now there must be a vertex of color 1 in B_5 which is unique for e_3. We now consider the edges e_4, e_5, e_6. These edges must be conflict-free 2-colorable with colors $2, 3$. The edges e_4, e_5, e_6 form the $\frac{r}{2}$-blow up of K_3 which is not conflict-free 2-colorable, a contradiction. □

3. Conflict-free coloring of hypergraphs with few edges

Having dealt with small cases, now we study the bounds for the conflict-free chromatic number for a general case. We start with a simple probabilistic fact we shall use later on.

Lemma 5. *Color a set T of t points, randomly, with s colors, so that each of s^t colorings is equally likely. Let $p_{t,s}$ be the probability that in the coloring no color appears exactly once and let $\hat{p}_{t,s}$ be the probability that in the coloring at most one color appears exactly once. Then*

$$p_{t,s} \le \left(\frac{2t}{s}\right)^{t/2}. \tag{1}$$

and

$$\hat{p}_{t,s} \le \left(\frac{8t}{s}\right)^{\lceil (t-1)/2 \rceil}. \tag{2}$$

Proof. Since both inequalities are proved in the same way, we shall only show the bound for $\hat{p}_{t,s}$. Let us color all elements of T, $|T| = t$, one by one. Note that we shall use at most $\lfloor t/2 \rfloor + 1 \le t$ colors. Furthermore, the set T' of the elements which are colored with a color which we have already used has at least $\lceil (t-1)/2 \rceil$ elements. Hence,

$$\hat{p}_{t,s} < 2^t \left(\frac{t}{s}\right)^{\lceil (t-1)/2 \rceil} \le \left(\frac{8t}{s}\right)^{\lceil (t-1)/2 \rceil},$$

where 2^t is an upper bound for the number of choices of $|T'|$. □

Now we can bound the $\chi_{CF}(\mathcal{H})$ for a general r-uniform hypergraph with m edges.

Theorem 6. *Let $\mathcal{H}$ be a r-uniform hypergraph with m edges and the maximum edge degree $D(\mathcal{H})$.*

(i) If $120 \ln D(\mathcal{H}) \le 2^{r/2}$, and $D(\mathcal{H})$ (and thus r) is large enough, then there exists a vertex coloring of $\mathcal{H}$ with $120 \ln D(\mathcal{H})$ colors such that each edge has at least one color appearing exactly once. In particular,

$$\chi_{CF}(\mathcal{H}) \le 120 \ln D(\mathcal{H}) \le 120 \ln m.$$

(ii) If $m \ge 2^{r/2}$, then $\chi_{CF}(\mathcal{H}) \le 4r(4m)^{2/(r+2)}$.

Proof. In order to show (i) we set $p = 1.34 \ln D(\mathcal{H})/r$, choose a subset $\hat{T}$ of vertices independently with probability p, and then color each vertex of $\hat{T}$ independently with one of $s = 120 \ln D(\mathcal{H})$ colors. Let A_e be the event that no color appears exactly once in the edge e. Then, by Lemma 5,

$$\mathbb{P}(A_e) \le \sum_{i=0}^{2.5\cdot 1.34 \ln D(\mathcal{H})} \binom{r}{i} p^i (1-p)^{r-i} \left(\frac{2i}{s}\right)^{i/2} + \sum_{i=2.5\cdot 1.34 \ln D(\mathcal{H})}^{r} \binom{r}{i} p^i (1-p)^{r-i}$$

$$\le \sum_{i=0}^{i_0} \binom{r}{i} p^i (1-p)^{r-i} \left(\frac{2i_0}{s}\right)^{i/2} + \sum_{i=i_0+1}^{r} \binom{r}{i} p^i (1-p)^{r-i},$$

where here and below $i_0 = 2.5 \cdot 1.34 \ln D(\mathcal{H})$.

Note that because for $i \geq i_0$ we have

$$\frac{\binom{r}{i+1}p^{i+1}(1-p)^{r-i-1}}{\binom{r}{i}p^i(1-p)^{r-i}} \leq \frac{r}{i}\frac{p}{1-p} \leq \frac{1}{2.5(1-p)} < \frac{1}{1.16}\,,$$

so the second sum can be bounded from above by a geometric series and consequently

$$\sum_{i=i_0+1}^{r}\binom{r}{i}p^i(1-p)^{r-i} \leq 7.25\binom{r}{i_0}p^{i_0}(1-p)^{r-i_0}$$

Since $\binom{r}{j} \leq (\frac{er}{j})^j$ and $(1-p)^{r-j} \leq (1-p)^r \leq (e^{-pr/j})^j$ we have

$$\begin{aligned}\mathbb{P}(A_e) &\leq \Big(1+\Big(\sqrt{\frac{2i_0}{s}}-1\Big)p\Big)^r + 7.25\Big(\frac{erp}{i_0}\cdot e^{-pr/i_0}\Big)^{i_0}\\ &\leq \exp(-0.76pr) + 7.25\exp\big(-0.79\cdot 1.34\ln D(\mathcal{H})\big)\\ &\leq D(\mathcal{H})^{-1.01} + 7.25D(\mathcal{H})^{-1.05} \leq 1/(4D(\mathcal{H}))\end{aligned}$$

for sufficiently large $D(\mathcal{H})$. Consequently, $D(\mathcal{H})\mathbb{P}(A_e) < 1/4$ and by Lovász Local lemma, there exists a conflict-free coloring of $\mathcal{H}$, so (i) follows.

Now let us set $s = 2r(4m)^{2/(r+2)}$ and $k = 2s$. We shall show that $\mathcal{H}$ has a conflict-free coloring with at most k colors. Let v be a vertex of maximum degree in $\mathcal{H}$. Reserve a color c for v and delete it along with all the edges containing it. Repeat this procedure and reserve a different color every time we delete a vertex of maximum degree in the remaining hypergraph. This procedure is repeated $k/2$ times. Let $\mathcal{H}_1$ denote hypergraph obtained by $k/2$ repetitions of this procedure. We consider the two following cases.

Case 1. $D(\mathcal{H}_1) < m^{r/(r+2)}$.

Color each vertex of $\mathcal{H}_1$ by a color chosen randomly among s colors. Let A_e be the event that no color appears exactly once in the edge e. From Lemma 5, $\mathbb{P}(A_e) < (2r/s)^{r/2}$. Thus,

$$4\cdot D(\mathcal{H}_1)\cdot\mathbb{P}(A_e) < 4\cdot m^{r/(r+2)}\cdot(2r/s)^{r/2} = 1.$$

Hence by Lovász Local Lemma, there exists a conflict-free coloring of $\mathcal{H}_1$ with $k/2$ colors. Together with the other $k/2$ colors, we have a conflict-free coloring of $\mathcal{H}$ with $k = 2s = 4r(4m)^{2/(r+2)}$ colors.

Case 2. $D(\mathcal{H}_1) \geq m^{r/(r+2)}$.

Note that since each time we have deleted a vertex of maximum degree in the remaining hypergraph, we have removed at least $\frac{D(\mathcal{H}_1)}{r} \geq \frac{m^{r/(r+2)}}{r}$ edges $k/2$ times. Thus, $m \geq km^{r/(r+2)}/(2r)$ which implies $k \leq 2rm^{2/(r+2)}$, which completes the proof of (ii). □

It is not hard to see that the bound given by Theorem 6(i) is tight up to a constant factor. Indeed, the following holds.

Proposition 7. *For all m and for all even r, there exists an r-uniform hypergraph $\mathcal{H}$ with m edges such that $\chi_{CF}(\mathcal{H}) > \frac{1}{2}(\log_2 m + 1)$.*

Proof. Consider the hypergraph $\mathcal{H}$ which is a $\frac{r}{2}$-blow up of K_n, where the blobs are B_i, $i \in \{1, 2, ...n\}$. Thus $m = \binom{n}{2}$. We claim that $k = \chi_{CF}(\mathcal{H}) > \frac{1}{2}\log_2 m + 1$. Let f be a conflict-free coloring of $\mathcal{H}$ with k colors. Let S_i be the set of colors that appear in blob B_i. There are $2^k - 1$ possibilities to choose a non-empty S_i. If $2^k - 1 < n$, then $S_i = S_j$ for some $i \neq j$, implying that f is not conflict-free, a contradiction. Hence $n < 2^k$, which implies $2^k > \sqrt{2m}$. Thus $k > \frac{1}{2}(1 + \log_2 m)$. □

To construct a matching bound for Theorem 6(ii), when m is much larger than r, is a harder task. Pach and Tardos [12] showed that if $\mathcal{H}$ is a r-uniform hypergraph with m edges, then $\chi_{CF}(\mathcal{H}) \leq rm^{2/(r+1)} \log m$, and they ask whether $\chi_{CF}(\mathcal{H}) \leq rm^{1/r} \log m$. We answer their question in the negative. More precisely, we show that if r is much smaller than m, then there exists r-uniform hypergraph $\mathcal{H}$ such that $\chi_{CF}(\mathcal{H}) \geq C_r m^{2/(r+2)}/\log m$. Let us start with a simple observation.

Observation 8. *Given any coloring f of an n element set in k colors, we can choose a family $\mathcal{A}_f$ of k disjoint sets such that each set in $\mathcal{A}_f$ has size $\lceil n/2k \rceil$ and is monochromatic.*

Proof. Consider the color classes $A_1, A_2, ...A_k$. For each color class A_i we partition it into subclasses B_{i_j} of size equal to $\lceil n/2k \rceil$ until we cannot anymore. The last subclass say $B_{i_{j'}}$ for some values of i will have size at most $n/2k$. Summing the sizes of these $B_{i_{j'}}$ we get at most $n/2$ vertices. The remaining at least $n/2$ vertices gives us a family of k sets such that each set in $\mathcal{A}_f$ has size $\lceil n/2k \rceil$ and is monochromatic. □

Lemma 9. *For a fixed k and for every even r, $r \leq 2k$, there exists an r-uniform hypergraph $\mathcal{H}$ with $m = \lceil (4(4e^2/r)^{r/2} k^{(r+2)/2} \log k \rceil$ edges such that $\chi_{CF}(\mathcal{H}) > k$.*

Proof. Consider a vertex set V of size $n = 4k$. We form a random r-uniform hypergraph with $m = (\lceil (4(4e^2/r)^{r/2} k^{(r+2)/2} \log k \rceil)$ edges by choosing m subsets $F_1, F_2, ..., F_m$ of V of size r randomly with equal probability and repetitions allowed and prove that with a positive probability it conflict-free chromatic number is larger that k.

Let f be any fixed k-coloring of V. By Observation 8, there exists a family $\mathcal{A}_f$ of k sets $\{A_1, A_2, ..., A_k\}$ such that each of these sets has size $\lceil n/2k \rceil$ and is monochromatic. So the probability that a given edge has a conflict is bounded from above by the probability that it has exactly 2 or 0 vertices from each of the sets in $\mathcal{A}_f$, which, in turn, can be bounded from above by $\binom{k}{r/2}\binom{\lceil n/(2k) \rceil}{2}^{r/2}\binom{n}{r}^{-1}$. Since

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{en}{k}\right)^k,$$

we get

$$\mathbb{P}(\text{edge } F_i \text{ has a conflict}) \geq \left(\frac{k}{r/2}\right)^{r/2} \left(\frac{n^2}{8k^2}\right)^{r/2} \left(\frac{r^2}{e^2 n^2}\right)^{r/2} = \left(\frac{r}{4e^2 k}\right)^{r/2}.$$

Consequently,

$$\begin{aligned}\mathbb{P}(f \text{ is a conflict-free coloring of } \mathcal{H}) &\leq \Big(1-\Big(\frac{r}{8e^2k}\Big)^{r/2}\Big)^m\\ &< \exp\Big(-m\Big(\frac{r}{8e^2k}\Big)^{r/2}\Big).\end{aligned}$$

There are k^n distinct colorings of $V(\mathcal{H})$, so

$$\begin{aligned}\mathbb{P}(\mathcal{H} \text{ is conflict-free colorable}) &< k^n \exp\Big(-m\Big(\frac{r}{4e^2k}\Big)^{r/2}\Big)\\ &\leq k^n \exp\Big(-m\Big(\frac{r}{4e^2k}\Big)^{r/2} + n\log k\Big).\end{aligned}$$

Since $m = \lceil (4(4e^2/r)^{r/2}k^{(r+2)/2}\log k\rceil$ and $n = 4k$, we infer that the probability that $\mathcal{H}$ is conflict-free colorable is strictly smaller than one. Hence there exists a hypergraph $\mathcal{H}$ with $m = \lceil (4(4e^2/r)^{r/2}k^{(r+2)/2}\log k\rceil$ edges such that $\chi_{CF}(\mathcal{H}) > k$. □

Remark: Let $m = \lceil (4(8e^2/r)^{r/2}k^{(r+2)/2}\log k\rceil$. Solving this for k, we get $k \sim C_r m^{2/(r+2)}/\log m$, where C_r is a function of r. Thus, Lemma 9 shows that for a given m and $r \leq C_r m^{2/(r+2)}/\log m$, there exists an r-uniform hypergraph $\mathcal{H}$ with m edges such that $\chi_{CF}(\mathcal{H}) > C_r m^{2/(r+2)}/\log m$.

4. Conflict-free coloring of simple hypergraphs

Although one can show that there exist simple r-uniform hypergraphs $\mathcal{H}$ with $m = C^r$ such that $\chi(\mathcal{H}) = \Theta(r)$, the second part of Theorem 6(ii) can be improved in the case of simple hypergraphs. Let us start with the following simple consequence of Lemma 5.

Lemma 10. *Let $r \leq k/8$ and let $\mathcal{H}$ be an r-uniform hypergraph. If $D(\mathcal{H}) < \frac{1}{4}(\frac{k}{8r})^{\lceil (r-1)/2\rceil}$ then there exists a vertex coloring of $\mathcal{H}$ with k colors such that each edge has at least two colors appearing exactly once.*

Proof. Let us consider a random k-coloring of $\mathcal{H}$ and let A_e be the event that the edge e has at most one color appearing exactly once. By Lemma 5, the probability of A_e, $\mathbb{P}(A_e) \leq (\frac{8r}{k})^{\lceil (r-1)/2\rceil}$. Now note that for a given edge e the event A_e is dependent on at most $D(\mathcal{H})$ other events $A_{e'}$. Thus, for $D(\mathcal{H}) < \frac{1}{4}(\frac{k}{8r})^{\lceil (r-1)/2\rceil}$, we have $4\cdot\mathbb{P}(A_e)\cdot D(\mathcal{H}) < 1$, and so by Lovász Local Lemma there exists a coloring where none of the events A_e occur. Consequently, there exists a coloring of $\mathcal{H}$ with k colors such that every edge has at least two colors appearing exactly once. □

Remark. By Lemma 7, for a given m, even if r is arbitrarily large (but even), there is an r-uniform hypergraph $\mathcal{H}$ with m edges and $\chi_{CF}(\mathcal{H}) > 0.5\log_2 m$. There is no similar statement for simple hypergraphs. Indeed, if the maximum edge degree of a simple r-uniform hypergraph $\mathcal{H}$ is less than r, then we can choose in each edge e a vertex v_e that belongs only to e. Then we color each v_e with 1, and every other vertex with 2. So, such a hypergraph has a conflict-free coloring with just 2 colors.

Theorem 11. *Let $r \le k/8$ and let $\mathcal{H}$ be an r-uniform simple hypergraph with m edges. If $m \le \frac{1}{16r(r-1)^2}(\frac{k}{8(r-1)})^{r-2}$, then $\chi_{CF}(\mathcal{H}) \le k$.*

Proof. Assume that $\chi_{CF}(\mathcal{H}) > k$. Let $\mathcal{H}_1$ be the hypergraph obtained from $\mathcal{H}$ by truncating each edge by a vertex of maximum possible degree. Observe that $\mathcal{H}_1$ is an $(r-1)$-uniform simple hypergraph and if f is a k-coloring of $\mathcal{H}_1$, then there exists an edge of $\mathcal{H}_1$ which has at most one color appearing exactly once, otherwise $\mathcal{H}$ would be conflict-free k-colorable. Now by Lemma 10, $D(\mathcal{H}_1) \ge \frac{1}{4}(\frac{k}{8(r-1)})^{\lceil (r-2)/2 \rceil}$. Thus $\mathcal{H}_1$ has a vertex of degree at least $D(\mathcal{H}_1)/(r-1)$. If $\mathcal{H}_1$ has a vertex of degree at least d, then by the way we truncated edges, we infer that $\mathcal{H}$ has at least d vertices of degree at least d and moreover, since $\mathcal{H}$ is simple, all these d vertices are distinct. Hence $\mathcal{H}$ has at least $D(\mathcal{H}_1)/(r-1)$ vertices of degree at least $D(\mathcal{H}_1)/(r-1)$. So by the degree-sum formula, $m \ge D(\mathcal{H}_1)^2/r(r-1)^2 > \frac{1}{16r(r-1)^2}(\frac{k}{8(r-1)})^{r-2}$. □

Note that if we solve the equation $m = \frac{1}{16r(r-1)^2}(\frac{k}{8(r-1)})^{r-2}$ with respect to k we get $k \sim C'_r m^{1/(r-2)}$ so, for large r, the upper bound for the conflict-free chromatic number for simple hypergraphs provided by Theorem 11 is roughly a square of the bound given by Theorem 6 for the general case. The following result shows that, at least for large r, this estimate is not very far from being optimal.

Lemma 12. *Let $r \le k$. Then, there exists an r-uniform simple hypergraph $\mathcal{H}$ with $(1+o(1))(4k\ln k)^2(\frac{4e^2k}{r})^r$ edges such that $\chi_{CF}(\mathcal{H}) > k$.*

Proof. We first construct an auxiliary $4k$-uniform simple hypergraph $\mathcal{H}_1$ as follows. Let q be a prime which will be decided later. The vertex set of $\mathcal{H}_1$ is $S = S_1 \cup ... \cup S_{4k}$ where all S_i are disjoint copies of $GF(q) = \{0, 1, ..., q-1\}$. The edges of $\mathcal{H}_1$ are $4k$-tuples $(x_1, ..., x_{4k}) \in S_1 \times ... \times S_{4k}$ that are solutions of the system of linear equations

$$\sum_{i=1}^{4k} i^j x_i = 0, \qquad j = 0, 1, ..., 4k-3, \tag{3}$$

over $GF(q)$.

For any arbitrary fixed pair of variables in (3), we have a $(4k-2) \times (4k-2)$ system of linear equations with Vandermond's determinant which has a unique solution over $GF(q)$. This means that $\mathcal{H}_1$ is $4k$-uniform simple hypergraph with $4kq$ vertices in which each vertex is contained in q edges, so $|E(\mathcal{H}_1)| = q^2$.

Now from each edge e of $\mathcal{H}_1$ we choose an r-subset A_e randomly and independently. Let $\mathcal{H}$ be the r-uniform simple hypergraph obtained from $\mathcal{H}_1$ by taking the subsets A_e as its edges. Our goal is to show that with a positive probability the conflict-free chromatic number of $\mathcal{H}$ is large.

To this end, let us fix a coloring f. Let A_e denote the event that the edge e has a conflict in the coloring f, and $p = \mathbb{P}(A_e)$. Arguing as in the proof of Lemma 9, one can show that

$$p \ge \left(\frac{r}{4e^2k}\right)^{r/2}.$$

Since the edges of $\mathcal{H}$ were chosen independently, the probability that f is a conflict-free coloring of $\mathcal{H}$ is $(1-p)^{q^2}$. Moreover, the total number of colorings is k^{4kq}, so the probability that there exists a conflict-free coloring of $\mathcal{H}$ with k colors is at most $k^{4kq} \cdot (1-p)^{q^2}$. This probability is less than 1, provided

$$k^{4kq} \cdot e^{-pq^2} < 1\,,$$

which, holds whenever

$$q > 4k \ln k \Big(\frac{4e^2 k}{r}\Big)^{r/2} \geq \frac{4k \ln k}{p}.$$

Now if we take the smallest prime such that $q > q_0 = 4k \ln k (\frac{4e^2 k}{r})^{r/2}$ to have an r-uniform simple hypergraph with q^2 edges and $\chi_{CF}(\mathcal{H}) > k$. It is well known (see, for instance, [10]) that one can take $q = (1+o(1))q_0$. Hence

$$|E(\mathcal{H})| = (1+o(1))(4k \ln k)^2 \Big(\frac{4e^2 k}{r}\Big)^r.$$

□

Finally, let us remark that if we take $k = r$ we get a simple r-uniform hypergraph $\mathcal{H}$ with $m = 2^{O(r)}$ edges such that $\chi(\mathcal{H}) > r = \Omega(\ln m)$, so Theorem 6(i) cannot be much improved in the case of simple hypergraphs, at least when m grows exponentialy with r.

Acknowledgement. We thank Wojtek Samotij for his helpful comments.

References

[1] N. Alon, Hypergraphs with high chromatic number, *Graphs and Combinatorics*, **1**, 387–389, 1985.

[2] N. Alon and S. Smorodinsky, Conflict-free colorings of shallow discs, in: *22nd Ann. ACM Symposium on Computational Geometry*, 2006, 41–43.

[3] A. Bar-Noy, P. Cheilaris, S. Olonetsky, and S. Smorodinsky, Online conflict-free colorings for hypergraphs. *Automata, languages and programming*, 219–230, Lecture Notes in Comput. Sci., 4596, Springer, Berlin, 2007.

[4] A. Bar-Noy, P. Cheilaris, and S. Smorodinsky, Deterministic conflict-free coloring for intervals: from offline to online. *ACM Trans. Algorithms* **4** (2008), no. 4, Art. 44, 18 pp.

[5] J. Beck, On 3-chromatic hypergraphs, *Discrete Math.* **24** (1978), 127-137.

[6] K. Chen, A. Fiat, H. Kaplan, M. Levy, J. Matoušek, E. Mossel, J. Pach, M. Sharir, S. Smorodinsky, U. Wagner, E. Welzl, Online conflict-free coloring for intervals, *SIAM J. Comput.* **36** (2006/07), 1342–1359.

[7] P. Erdős and L. Lovász, Problems and results on 3-chromatic hypergraphs and some related questions, In *Infinite and Finite Sets*, A. Hajnal, R. Rado, V.T. Sós, editors, Colloq. Math. Soc. J. Bolyai 11, North Holland, Amsterdam, (1975), 609–627.

[8] G. Even, Z. Lotker, D. Ron and S. Smorodinsky, Conflict-free colorings of simple geometric regions with applications to frequency assignment in cellular networks, *SIAM J.Comput.*, **33** (2003), 94–136.

[9] S. Har-Peled and S. Smorodinsky, Conflict-free coloring of points and simple regions in the plane. *Discrete Comput. Geom.* **34** (2005), 47–70.

[10] M.N. Huxley and H. Iwaniec, Bombieri's theorem in short intervals, *Mathematika*, **22** (1975), 188–194.

[11] A.V. Kostochka, M. Kumbhat and V. Rödl, Coloring uniform hypergraphs with small edge degrees, to appear in: *Fete of Combinatorics*, Bolyai Society Mathematical Studies.

[12] J. Pach and G. Tardos, Conflict-free colorings of graphs and hypergraphs, *Combin. Probab. Comput*, **18** (2009), no.5, 819–834.
[13] J. Pach and G. Tóth, Conflict-free colorings. *Discrete and computational geometry*, 665–671, Algorithms Combin., 25, Springer, Berlin, 2003.
[14] J. Radhakrishnan and A. Srinivasan, Improved bounds and algorithms for hypergraph two-coloring, *Random Structures and Algorithms*, **16**, (2000), 4–32.
[15] J. Spencer, Coloring n-sets red and blue, *J. Comb.Theory Ser. A*, **30**(1981), 112-113.

University of Illinois, Urbana, IL, 61801 and Institute of Mathematics, Novosibirsk, 630090, Russia.
E-mail address: kostochk@math.uiuc.edu.

Department of Mathematics, University of Illinois, Urbana, IL, 61801, USA.
E-mail address: kumbhat2@uiuc.edu.

Faculty of Mathematics and Computer Science, Adam Mickiewicz University, ul. Umultowska 87, 61-614 Poznań, Poland
E-mail address: tomasz@amu.edu.pl