6. \( f(x, y) = x \sin(xy) \Rightarrow f_x(x, y) = x \cos(xy) \cdot y + \sin(xy) = xy \cos(xy) + \sin(xy) \) and 
\( f_y(x, y) = x \cos(xy) \cdot x = x^2 \cos(xy). \) If \( \mathbf{u} \) is a unit vector in the direction of \( \theta = \frac{\pi}{3}, \) then from Equation 6,

\[
D_{\mathbf{u}} f(2, 0) = f_x(2, 0) \cos \frac{\pi}{3} + f_y(2, 0) \sin \frac{\pi}{3} = 0 + 4 \left( \frac{\sqrt{3}}{2} \right) = 2 \sqrt{3}.
\]

24. \( f(x, y, z) = \frac{x + y}{z} \Rightarrow \nabla f(x, y, z) = \left( \frac{1}{z}, \frac{-1}{z}, \frac{-x + y}{z^2} \right), \nabla f(1, 1, -1) = (1, -1, 2). \) Thus the maximum rate of change is \( |\nabla f(1, 1, -1)| = \sqrt{1 + 1 + 4} = \sqrt{6} \) in the direction \( (1, -1, -2). \)

39. Let \( F(x, y, z) = 2(x - 2)^2 + (y - 1)^2 + (z - 3)^2. \) Then \( 2(x - 2)^2 + (y - 1)^2 + (z - 3)^2 = 10 \) is a level surface of \( F. \)

\[
F_x(x, y, z) = 4(x - 2) \Rightarrow F_x(3, 3, 5) = 4, F_y(x, y, z) = 2(y - 1) \Rightarrow F_y(3, 3, 5) = 4, \text{ and } \]
\[ F_z(x, y, z) = 2(z - 3) \Rightarrow F_z(3, 3, 5) = 4. \]
(a) Equation 19 gives an equation of the tangent plane at \( (3, 3, 5) \) as \( 4(x - 3) + 4(y - 3) + 4(z - 5) = 0 \Longleftrightarrow 4x + 4y + 4z = 44 \text{ or equivalently } x + y + z = 11. \]
(b) By Equation 20, the normal line has symmetric equations \( \frac{x - 3}{4} = \frac{y - 3}{4} = \frac{z - 5}{4} \) or equivalently \( x - 3 = y - 3 = z - 5. \) Corresponding parametric equations are \( x = 3 + t, y = 3 + t, z = 5 + t. \)

44. \( F(x, y, z) = yz - \ln(x + z) \Rightarrow \nabla F(x, y, z) = \left( \frac{-1}{x+z}, z, y - \frac{1}{x+z} \right) \text{ and } \nabla F(0, 0, 1) = (-1, 1, -1). \)

(a) \( (-1)(x - 0) + (1)(y - 0) - 1(z - 1) = 0 \text{ or } x - y + z = 1 \)

(b) Parametric equations are \( x = -t, \ y = t, \ z = 1 - t \) and symmetric equations are \( \frac{x}{-1} = \frac{y}{1} = \frac{z - 1}{-1} \text{ or } -x = y = 1 - z. \)

47. \( f(x, y) = xy \Rightarrow \nabla f(x, y) = (y, x), \nabla f(3, 2) = (2, 3) \text{ and } \nabla f(3, 2) \) is perpendicular to the tangent line, so the tangent line has equation

\[ \nabla f(3, 2) \cdot (x - 3, y - 2) = 0 \Rightarrow (2, 3) \cdot (x - 3, y - 2) = 0 \Rightarrow 2(x - 3) + 3(y - 2) = 0 \text{ or } 2x + 3y = 12. \]

48. \( g(x, y) = x^2 + y^2 - 4x \Rightarrow \nabla g(x, y) = (2x - 4, 2y), \nabla g(1, 2) = (-2, 4). \) \( \nabla g(1, 2) \) is perpendicular to the tangent line, so the tangent line has equation \( \nabla g(1, 2) \cdot (x - 1, y - 2) = 0 \Rightarrow (-2, 4) \cdot (x - 1, y - 2) = 0 \Rightarrow -2(x - 1) + 4(y - 2) = 0 \Longleftrightarrow -2x + 4y = 6 \text{ or equivalently } -x + 2y = 3. \)
52. Let \( F(x, y, z) = x^2 + y^2 - z \), then the paraboloid \( y = x^2 + z^2 \) is a level surface of \( F \). \( \nabla F(x, y, z) = (2x, -1, 2z) \) is a normal vector to the surface at \((x, y, z)\) and so it is a normal vector for the tangent plane there. The tangent plane is parallel to the plane \( x + 2y + 3z = 1 \) when the normal vectors of the planes are parallel, so we need a point \((x_0, y_0, z_0)\) on the paraboloid where \((2x_0, -1, 2z_0) = k(1, 2, 3)\). Comparing \( y \)-components we have \( k = -\frac{1}{2} \), so

\[
(2x_0, -1, 2z_0) = \left( -\frac{1}{2}, -1, -\frac{3}{2} \right)
\]

and

\[
x_0 = -\frac{1}{4}, 2z_0 = -\frac{3}{2} \quad \Rightarrow \quad z_0 = -\frac{3}{4}.
\]

Then

\[
y_0 = x_0^2 + z_0^2 = \left( -\frac{1}{4} \right)^2 + \left( -\frac{3}{4} \right)^2 = \frac{5}{8}
\]

and the point is \(( -\frac{1}{4}, \frac{5}{8}, -\frac{3}{4} )\).

59. If \( f(x, y, z) = z - x^2 - y^2 \) and \( g(x, y, z) = 4x^2 + y^2 + z^2 \), then the tangent line is perpendicular to both \( \nabla f \) and \( \nabla g \) at \((-1, 1, 2)\). The vector \( \mathbf{v} = \nabla f \times \nabla g \) will therefore be parallel to the tangent line.

We have \( \nabla f(x, y, z) = (-2x, -2y, 1) \) \( \Rightarrow \nabla f(-1, 1, 2) = (2, -2, 1) \), and \( \nabla g(x, y, z) = (8x, 2y, 2z) \) \( \Rightarrow \)

\[
\nabla g(-1, 1, 2) = (-8, 2, 4).
\]

Hence \( \mathbf{v} = \nabla f \times \nabla g = \begin{vmatrix} i & j & k \\ 2 & -2 & 1 \\ -8 & 2 & 4 \end{vmatrix} = -10i - 16j - 12k.
\]

Parametric equations are: \( x = -1 - 10t \), \( y = 1 - 16t \), \( z = 2 - 12t \).

7. \( f(x, y) = x^4 + y^4 - 4xy + 2 \) \( \Rightarrow \ f_x = 4x^3 - 4y, f_y = 4y^3 - 4x, \)

\( f_{xx} = 12x^2, f_{xy} = -4, f_{yy} = 12y^2 \). Then \( f_x = 0 \) implies \( y = x^3 \),

and substitution into \( f_y = 0 \) \( \Rightarrow \ x = y^3 \) gives \( x^9 - x = 0 \)

\( x(x^8 - 1) = 0 \) \( \Rightarrow x = 0 \) or \( x = \pm 1 \). Thus the critical points are \((0, 0)\), \((1, 1)\), and \((-1, -1)\). Now \( D(0, 0) = 0 \cdot 0 - (-4)^2 = -16 < 0 \),

so \((0, 0)\) is a saddle point. \( D(1, 1) = (12)(12) - (-4)^2 > 0 \) and

\( f_{xx}(1, 1) = 12 > 0 \), so \( f(1, 1) = 0 \) is a local minimum. \( D(-1, -1) = (12)(12) - (-4)^2 > 0 \) and

\( f_{xx}(-1, -1) = 12 > 0 \), so \( f(-1, -1) = 0 \) is also a local minimum.

12. \( f(x, y) = xy + \frac{1}{x} + \frac{1}{y} \) \( \Rightarrow \ f_x = y - \frac{1}{x^2}, f_y = x - \frac{1}{y^2}, f_{xx} = \frac{2}{x^3}, \)

\( f_{xy} = f_{yx} = \frac{2}{y^3} \). Then \( f_x = 0 \) implies \( y = \frac{1}{x^2} \) and \( f_y = 0 \) implies

\( x = \frac{1}{y^2} \) \( \Rightarrow \)

Substituting the first equation into the second gives

\( x = \frac{1}{(1/x^2)} \) \( \Rightarrow x = x^4 \) \( \Rightarrow x(x^3 - 1) = 0 \) \( \Rightarrow \) \( x = 0 \) or \( x = 1 \).

\( f \) is not defined when \( x = 0 \), and when \( x = 1 \) we have \( y = 1 \), so the only critical point is \((1, 1)\).

\( D(1, 1) = (2)(2) - 1^2 = 3 > 0 \) and \( f_{xx}(1, 1) = 2 > 0 \), so \( f(1, 1) = 3 \) is a local minimum.
13. \( f(x, y) = e^x \cos y \implies f_x = e^x \cos y, f_y = -e^x \sin y. \)

Now \( f_x = 0 \) implies \( \cos y = 0 \) or \( y = \frac{\pi}{2} + n\pi \) for \( n \) an integer.

But \( \sin \left( \frac{\pi}{2} + n\pi \right) \neq 0 \), so there are no critical points.

16. \( f(x, y) = e^y (y^2 - x^3) \implies f_x = -2xe^y, f_y = (2y + y^2 - x^3)e^y, \)
\( f_{xx} = -2e^y, f_{xy} = -2xe^y, f_{yy} = (2 + 4y + y^2 - x^3)e^y. \) Then \( f_x = 0 \)
implies \( x = 0 \) and substituting into \( f_y = 0 \) gives \( (2y + y^2)e^y = 0 \implies y(2 + y) = 0 \implies y = 0 \text{ or } y = -2, \) so the critical points are \((0, 0)\) and \((0, -2)\). \( D(0, 0) = (-2)(2) - (0)^2 = -4 < 0 \) so \((0, 0)\) is a saddle point.
\( D(0, -2) = (-2e^{-2})(-2e^{-2}) - (0)^2 = 4e^{-4} > 0 \) and \( f_{xx}(0, -2) = -2e^{-2} < 0, \) so \( f(0, -2) = 4e^{-2} \) is a local maximum.

18. \( f(x, y) = \sin x \cdot \sin y \implies f_x = \cos x \cdot \sin y, f_y = \sin x \cdot \cos y, f_{xx} = -\sin x \cdot \sin y, f_{yy} = \cos x \cdot \cos y, \)
\( f_{xy} = -\sin x \cdot \sin y. \) Here we have \( -\pi < x < \pi \) and \( -\pi < y < \pi, \) so \( f_x = 0 \) implies \( \cos x = 0 \) or \( \sin y = 0. \) If \( \cos x = 0 \)
then \( x = -\frac{\pi}{2} \) or \( \frac{\pi}{2}, \) and if \( \sin y = 0 \) then \( y = 0. \) Substituting \( x = \pm \frac{\pi}{2} \) into \( f_y = 0 \) gives \( \cos y = 0 \implies y = -\frac{\pi}{2} \) or \( \frac{\pi}{2}, \) and
substituting \( y = 0 \) into \( f_y = 0 \) gives \( \sin x = 0 \implies x = 0. \) Thus the critical points are \(( -\frac{\pi}{2}, \pm \frac{\pi}{2} ), ( \frac{\pi}{2}, \pm \frac{\pi}{2} ), \) and \((0, 0)\).
\( D(0, 0) = -1 < 0 \) so \((0, 0)\) is a saddle point.
\( D(-\frac{\pi}{2}, \pm \frac{\pi}{2}) = D(\frac{\pi}{2}, \pm \frac{\pi}{2}) = 1 > 0 \) and 
\( f_{xx}(\frac{\pi}{2}, \pm \frac{\pi}{2}) = f_{xx}(\frac{\pi}{2}, \pm \frac{\pi}{2}) = -1 < 0 \) while 
\( f_{xx}(\frac{\pi}{2}, \frac{\pi}{2}) = f_{xx}(\frac{\pi}{2}, -\frac{\pi}{2}) = 1 > 0, \) so \( f(\frac{\pi}{2}, \frac{\pi}{2}) = f(\frac{\pi}{2}, -\frac{\pi}{2}) = 1 \) are local maxima and \( f(-\frac{\pi}{2}, \frac{\pi}{2}) = f(-\frac{\pi}{2}, -\frac{\pi}{2}) = 1 \) are local minima.