29. Since $f$ is a polynomial, it is continuous on $D$, so an absolute maximum and minimum exist. Here $f_x = 4, f_y = -5$ so there are no critical points inside $D$. Thus the absolute extrema must both occur on the boundary. Along $L_1: x = 0$ and $f(0, y) = 1 - 5y$ for $0 \leq y \leq 3$, a decreasing function in $y$, so the maximum value is $f(0, 0) = 1$ and the minimum value is $f(0, 3) = -14$. Along $L_2: y = 0$ and $f(x, 0) = 1 + 4x$ for $0 \leq x \leq 2$, an increasing function in $x$, so the minimum value is $f(0, 0) = 1$ and the maximum value is $f(2, 0) = 9$. Along $L_3: y = -\frac{3}{2}x + 3$ and $f(x, -\frac{3}{2}x + 3) = \frac{23}{2}x - 14$ for $0 \leq x \leq 2$, an increasing function in $x$, so the minimum value is $f(0, 3) = -14$ and the maximum value is $f(2, 0) = 9$. Thus the absolute maximum of $f$ on $D$ is $f(2, 0) = 9$ and the absolute minimum is $f(0, 3) = -14$.

31. $f_x(x, y) = 2x + 2xy, f_y(x, y) = 2y + x^2$, and setting $f_x = f_y = 0$ gives $(0, 0)$ as the only critical point in $D$, with $f(0, 0) = 4$.

On $L_1: y = -1, f(x, -1) = 5$, a constant.

On $L_2: x = 1, f(1, y) = y^2 + y + 5$, a quadratic in $y$ which attains its maximum at $(1, 1), f(1, 1) = 7$ and its minimum at $(1, -\frac{1}{2}), f(1, -\frac{1}{2}) = \frac{19}{4}$.

On $L_3: f(x, 1) = 2x^2 + 5$ which attains its maximum at $(-1, 1)$ and $(1, 1)$ with $f(\pm1, 1) = 7$ and its minimum at $(0, 1), f(0, 1) = 5$.

On $L_4: f(-1, y) = y^2 + y + 5$ with maximum at $(-1, 1), f(-1, 1) = 7$ and minimum at $(-1, -\frac{1}{2}), f(-1, -\frac{1}{2}) = \frac{19}{4}$.

Thus the absolute maximum is attained at both $(\pm1, 1)$ with $f(\pm1, 1) = 7$ and the absolute minimum on $D$ is attained at $(0, 0)$ with $f(0, 0) = 4$.

35. $f_x(x, y) = 6x^2$ and $f_y(x, y) = 4y^2$. And so $f_x = 0$ and $f_y = 0$ only occur when $x = y = 0$. Hence, the only critical point inside the disk is at $x = y = 0$ where $f(0, 0) = 0$. Now on the circle $x^2 + y^2 = 1$, $y = 1 - x^2$ so let $g(x) = f(x, y) = 2x^3 + (1 - x^2)^2 = x^4 + 2x^3 - 2x^2 + 1, -1 \leq x \leq 1$. Then $g'(x) = 4x^3 + 6x^2 - 4x = 0 \Rightarrow x = 0, -2, \text{ or } 2$. $f(0, \pm1) = g(0) = 1, f\left(\frac{1}{2}, \pm\sqrt{2}\right) = g\left(\frac{1}{2}\right) = \frac{12}{16}$, and $(-2, -3)$ is not in $D$. Checking the endpoints, we get $f(-1, 0) = g(-1) = -2$ and $f(1, 0) = g(1) = 2$. Thus the absolute maximum and minimum of $f$ on $D$ are $f(1, 0) = 2$ and $f(-1, 0) = -2$.

Another method: On the boundary $x^2 + y^2 = 1$ we can write $x = \cos \theta, y = \sin \theta$, so $f(\cos \theta, \sin \theta) = 2\cos^3 \theta + \sin^4 \theta, 0 \leq \theta \leq 2\pi$. 
41. Let $d$ be the distance from the point $(4, 2, 0)$ to any point $(x, y, z)$ on the cone, so $d = \sqrt{(x - 4)^2 + (y - 2)^2 + z^2}$ where $z^2 = x^2 + y^2$, and we minimize $d^2 = (x - 4)^2 + (y - 2)^2 + x^2 + y^2 = f(x, y)$. Then $f_x(x, y) = 2(x - 4) + 2x = 4x - 8$, $f_y(x, y) = 2(y - 2) + 2y = 4y - 4$, and the critical points occur when $f_x = 0 \implies x = 2$, $f_y = 0 \implies y = 1$. Thus the only critical point is $(2, 1)$. An absolute minimum exists (since there is a minimum distance from the cone to the point) which must occur at a critical point, so the points on the cone closest to $(4, 2, 0)$ are $(2, 1, \pm \sqrt{5})$.

43. $x + y + z = 100$, so maximize $f(x, y) = xy(100 - x - y)$. $f_x = 100y - 2xy - y^2$, $f_y = 100x - x^2 - 2xy$, $f_{xx} = -2y$, $f_{yy} = -2x$, $f_{xy} = 100 - 2x - 2y$. Then $f_x = 0$ implies $y = 0$ or $y = 100 - 2x$. Substituting $y = 0$ into $f_y = 0$ gives $x = 0$ or $x = 100$ and substituting $y = 100 - 2x$ into $f_y = 0$ gives $3x^2 - 100x = 0$ so $x = 0$ or $\frac{100}{3}$. Thus the critical points are $(0, 0)$, $(100, 0)$, $(0, 100)$ and $(\frac{100}{3}, \frac{100}{3})$.

$D(0, 0) = D(100, 0) = D(0, 100) = -10,000$ while $D(\frac{100}{3}, \frac{100}{3}) = 10,006$ and $f_{xx}(\frac{100}{3}, \frac{100}{3}) = -\frac{200}{3} < 0$. Thus $(0, 0)$, $(100, 0)$ and $(0, 100)$ are saddle points whereas $f(\frac{100}{3}, \frac{100}{3})$ is a local maximum. Thus the numbers are $x = y = z = \frac{100}{3}$. 