Math 242
Final Exam
Fall, 2005

Name: ____

Gauss−Green formula: $\oint_C m[x, y]\, dx + n[x, y]\, dy = \int\int_R \partial_x n[x, y] - \partial_y m[x, y]\, dx\, dy$

True or False. If false, explain why.

1) __________ The quantity X×Y is a vector.

2) __________ If X.Y = 0, then either X = 0 or Y = 0.

3) Two legitimate parameterizations of the circle C

$x^2 + y^2 = 16$ are:

Parameterization 1:

$\{x[t], y[t]\} = \{4\, \mathrm{Sin}[t], 4\, \mathrm{Cos}[t]\}$ with $0 \le t \le 2\pi$.

Parameterization 2:

$\{x[t], y[t]\} = \{4\, \mathrm{Cos}[t], 4\, \mathrm{Sin}[t]\}$ with $0 \le t \le 2\pi$.

Which of these two parameterizations do you use to calculate

$\oint_C -y\, dx + x\, dy$?

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Calculate by hand

$\oint_C -y\, dx + x\, dy$.

What does the result tell you about the net flow of

Field1[x, y] = {−y, x}

along this circle?

What does the result tell you about the net flow
Field2[x, y] = {x, y}
across this circle?

4) Are all path integrals path independent? Explain why or why not.

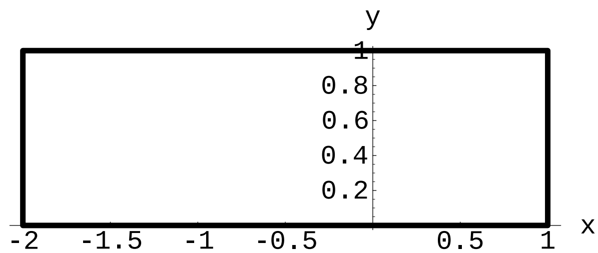
5) What are the conditions for which you would use the Gauss−Green formula?

What sort of problems does it make easier to calculate?

Fill in the blank, chose the appropriate word from the list:
6) If X.Y < 0, then is the push of X in the direction of Y is ____________________ Y ? (with or against)

7) Suppose f[x_0, y_0] > f[x, y] for all other {x, y} near {x_0, y_0}. If you center a small circle C at {x_0, y_0}, then you expect that the net flow of gradf[x, y] across C is from __________________. (inside to outside or outside to inside)

Answer the questions: Show work for full credit.
8) Here’s the rectang1e R with corners at
{−2, 0}, {−2, 1}, {1, 1}, and {1, 0}:



Use a 2D integral to measure the net flow of the vector field

Field[x, y] = $\{x^3 + y, x - y\}$

across the boundary curve C of this rectangle.

9) Go with f[x, y, z] = $x\,y\,e^z$.
In which direction should you leave the point {1, 2, 0} to get the greatest possible initial increase of f[x, y, z]?

10) How do you use the gradient to find the highest crest or the deepest dip on the plot of a surface z = f[x, y].

11) You are given:
→ A vector field Field[x, y] = {m[x, y], n[x, y]},
→ a closed curve C (like a deformed circle), and
→ a counterclockwise parameterization of C,

P[t] = {x[t], y[t]},

with P[t] sweeping out C exactly one time as t advances from tlow to thigh.
With these ingredients, you can make the following seven measurements:

i) $\oint_C$ Field.outerunitnormal ds,

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ii) $\oint_C m[x, y]\,dx + n[x, y]\,dy$,

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iii) $\int_{tlow}^{thigh} \text{Field}[x[t], y[t]].\{x'[t], y'[t]\}\, dt,$

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iv) $\oint_C \text{Field}.dP,$

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v) $\oint_C -n[x, y]\, dx + m[x, y]\, dy,$

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vi) $\oint_C \text{Field.unittan}\, ds,$

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vii) $\int_{tlow}^{thigh} \text{Field}[x[t], y[t]].\{y'[t], -x'[t]\}\, dt.$

Some of these measurements measure the net flow of Field[x, y] along C, and some measure the net flow of Field[x, y] across C.
Which are which?

12) You are faced with a hand calculation of

$$\int\int_R (x^2 + y^2)\, dx\, dy,$$

where R is the two−dimensional region consisting of everything inside and on the circle

$$x^2 + y^2 = 4.$$

Switch paper by using polar coordinates

x = r Cos[t], and
y = r Sin[t],

and then calculate the integral.

13) You are faced with a hand calculation of

$$\int\int_R x\, dx\, dy,$$

where R is the two−dimensional region consisting of everything bounded by the lines

$x + y = 0$, $x + y = 2$,

$x - y = 1$, and $x - y = 2$.

This is just a little nasty, because doing it without going to better paper involves solving some equations, and then doing two separate integrals.

Switch to paper on which you can evaluate this with one sweet integral and then calculate its value.

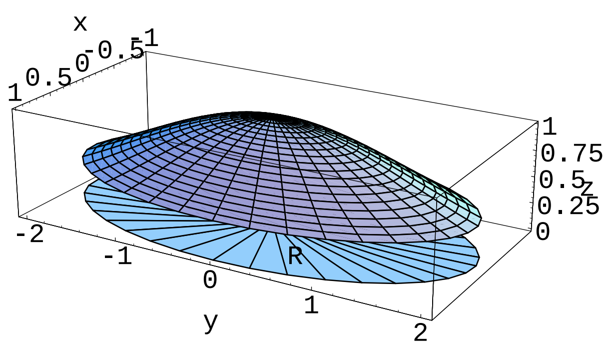
14) Here's a plot of the part of the surface

$$z = e^{-(x^2 + (y/2)^2)}$$

above the everything inside and on the ellipse

$$x^2 + (y/2)^2 = 1$$

in the xy−plane:



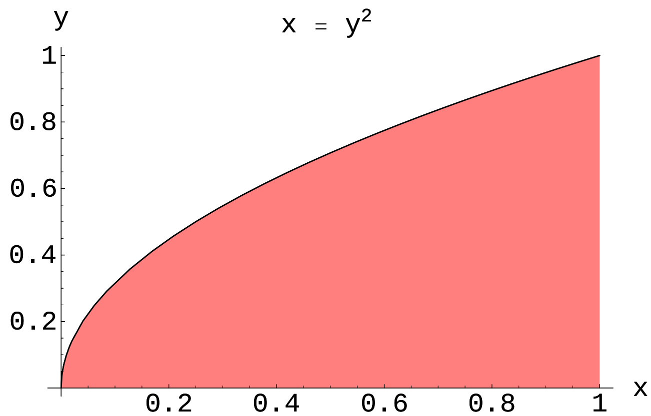
By hand calculation, measure the volume,

$$\int\int_R e^{-(x^2 + (y/2)^2)}\, dx\, dy,$$

of the solid whose top skin is the surface plotted above, and whose base is everything on the xy−plane directly below this surface, by transforming to more favorable paper.

Remember that $\text{Sin}[t]^2 + \text{Cos}[t]^2 = 1$.

15) Here's a nice little region in the xy−plane.



a) Fill in the boxes with the appropriate values to calculate the integrals.

$$\int_{\square}^{\square} \int_{\square}^{\square} f[x] \, dy \, dx =$$

$$\int_{\square}^{\square} \int_{\square}^{\square} f[x] \, dx \, dy$$

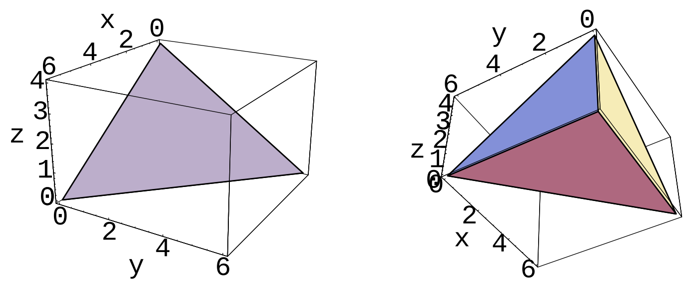
b) What does $\int\int_R 1 \, dA$ measure?

c) What does $\int\int_R (x^2 + y^3)\, dA$ measure?

16) R is the solid whose boundary surfaces are the xy−plane, the yz−plane, the xz−plane, and the plane

$$2x + 2y + 3z = 12.$$

Here are two looks at R:



The viewpoint on the left is the usual CMView;
the viewpoint on the right is a look from underneath.

Set up, but **do NOT EVALUATE** the integral to calculate

$$\int\int\int_R x\, dx\, dy\, dz.$$

17) Set numbers r*, slow, shigh, tlow, and thigh so that the xyz−points

{x[s, t], y[s, t], z[s, t]} =

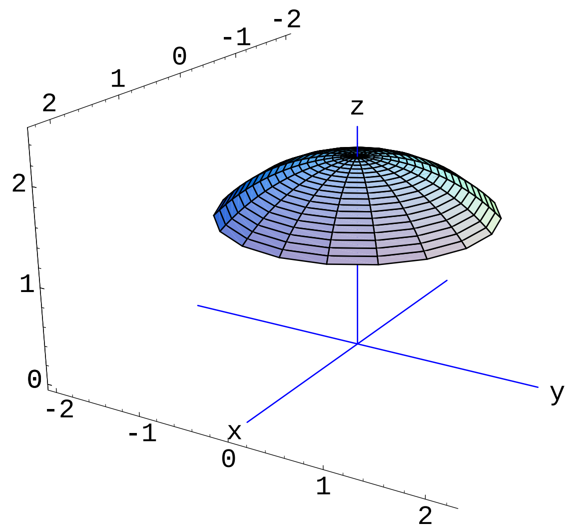
{r* Sin[s] Cos[t], r* Sin[s] Sin[t], r* Cos[s]}

with slow ≤ s ≤ shigh and tlow ≤ t ≤ thigh describe the part of the sphere

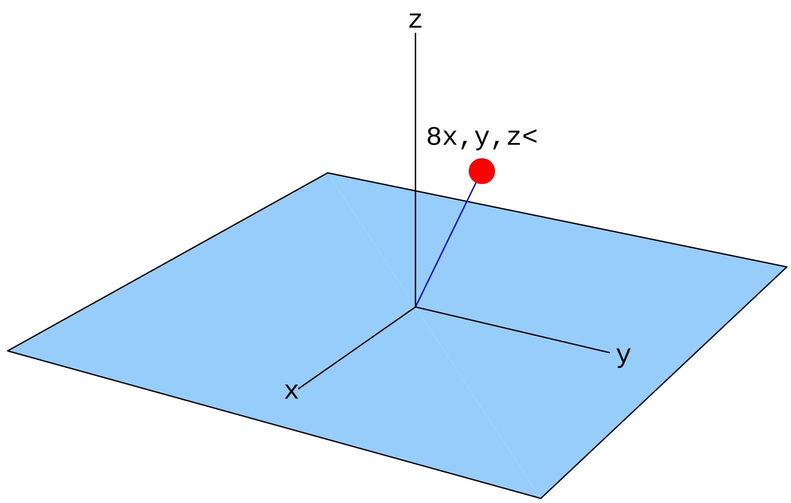
$$x^2 + y^2 + z^2 = 25$$

consisting of those points with $x \geq 0$.

18) Set numbers slow, shigh, tlow, and thigh so that the xyz−points
{x[s, t], y[s, t], z[s, t]} =
{2 Sin[s] Cos[t], 2 Sin[s] Sin[t], 2 Cos[s]}
with slow ≤ s ≤ shigh and tlow ≤ t ≤ thigh describe something like this:



19) You can specify a point in three dimensions as usual by its three coordinates {x, y, z}.
Another way to specify a point is to plant the x−, y−, and z−axes at {0, 0, 0}, and then run a stick from {0, 0, 0} to the point:



The spherical coordinates {r, s, t} of the plotted point are specified by the equations

$x = r \operatorname{Sin}[s] \operatorname{Cos}[t]$,

$y = r \operatorname{Sin}[s] \operatorname{Sin}[t]$, and

$z = r \operatorname{Cos}[s]$.

Take out a pencil and label the meaning of the measurements r, s, and t on the plot above.

20) Use spherical coordinates to pull off a very quick hand calculation of

$$\int\int\int_{R_{xyz}} (x^2 + y^2 + z^2)\, dx\, dy\, dz$$

where R_{xyz} is everything inside and on the sphere

$$x^2 + y^2 + z^2 = 4^2$$