Solutions to Exam 2

1. $\nabla f = \langle 2xy, x^2, 2ze^{z^2}\rangle$. $\nabla f(P) = \langle -4, 1, 2e\rangle$
$D_{\vec{u}} f(P) = \nabla f(P) \cdot \frac{\vec{v}}{|\vec{v}|} = \langle -4, 1, 2e\rangle \cdot \frac{\langle 3,1,2\rangle}{\sqrt{14}} = \frac{4e-13}{\sqrt{14}}$.
The most rapid **decrease** is in the direction $-\nabla f = \langle 4, -1, -2e\rangle$.

2. The boundary of the region over which we integrate is the intersection of the two surfaces. A quick look at the surfaces in question tells us that polar is the easier way to integrate with. Set $4r^2 = 20 - r^2$, so $r = 2$. Thus the volume is:

$$V = \int_0^{2\pi} \int_0^2 ((20 - r^2) - 4r^2) r \, dr \, d\theta = \int_0^{2\pi} \int_0^2 (20r - 5r^3) \, dr \, d\theta = 40\pi.$$

3. $\nabla F = \langle ze^{xz}, 3y^2z^2, 2y^3z + xe^{xz}\rangle$. $\nabla F(P) = \langle e^2, 3, -2 + 2e^2\rangle$, which is the normal vector of the plane. Therefore the equation of the plane is:

$$e^2(x-2) + 3(y+1) + (2e^2 - 2)(z-1) = 0.$$

4. By the chain rule $\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$. So:

$$\frac{\partial f}{\partial s} = \Big((3x^2z^2)\cos(x^3z^2 + y^5)te^{st+t^2}\Big) + \left(5y^4\cos(x^3z^2+y^5)\frac{1}{s}\right) + \Big(2x^3z\cos(x^3z^2+y^5)\sec^2(s+t)\Big).$$

5. $f(x,y) = x^2 + \frac{y^2}{4}$ and $g(x,y) = 3x^2 + y - 4$, so $\nabla f = \langle 2x, y/2\rangle$ and $\nabla g = \langle 6x, 1\rangle$. So we have the system:

$$\begin{cases} 3x^2 + y = 4 \\ 2x = 6\lambda x \\ \frac{y}{2} = \lambda \end{cases}$$

The second equation tells us either $x = 0$ or $\lambda = \frac{1}{3}$. If $x = 0$, then the first equation tells us that $y = 4$, so $f(0,4) = 4$ is a possibility for the minimum. If $\lambda = \frac{1}{3}$, then the third equation tells us that $y = \frac{2}{3}$, and the first equation tells us that $x = \pm\frac{\sqrt{10}}{3}$. Plugging these two solutions into f, we get $f\left(\pm\frac{\sqrt{10}}{3}, \frac{2}{3}\right) = \frac{11}{9}$. So of the three solutions, the absolute minimum is $\frac{11}{9}$ at

$\left(\pm\frac{\sqrt{10}}{3}, \frac{2}{3}\right)$.

6. a)

b) (i)

$$I = \int_{-6}^{-3} \int_{-\sqrt{x+6}}^{\sqrt{x+6}} f(x,y)\,dx\,dy + \int_{-3}^{0} \int_{-\sqrt{-x}}^{\sqrt{-x}} f(x,y)\,dx\,dy$$

(ii)

$$I = \int_{-\sqrt{3}}^{\sqrt{3}} \int_{y^2-6}^{-y^2} f(x,y)\,dx\,dy$$

7. We solve the system:

$$\begin{cases} f_x = -6x^2 + 24 = 0 \\ f_y = 2y + 2 = 0 \end{cases}$$

The first equation tells us that $x = \pm 2$ and the second tells us that $y = -1$, so there are two critical points $(\pm 2, -1)$. For the second derivative test, we find $f_{xx} = -12x$, $f_{xy} = 0$, $f_{yy} = 2$, so

$$\Delta(x,y) = \begin{vmatrix} -12x & 0 \\ 0 & 2 \end{vmatrix} = -24x.$$

Since $\Delta(-2,-1) > 0$ and $f_{xx}(-2,-1) > 0$, $(-2,-1)$ is a **local** minimum.
Since $\Delta(2,-1) < 0$, $(2,-1)$ is a saddle point.

Since $\lim\limits_{x\to\infty} f(x,y) = -\infty$, $f(x,y)$ has no absolute minimum.
Since $\lim\limits_{x\to-\infty} f(x,y) = \infty$, $f(x,y)$ has no absolute maximum.