Exam 3 Solutions

1.

$$\int_0^1 \int_x^1 \sqrt{1+\sin^2 x + 8y^4}\, dy\, dx \text{ OR } \int_0^1 \int_0^y \sqrt{1+\sin^2 x + 8y^4}\, dx\, dy$$

2.

$$\begin{aligned}\iiint_T (x^2+y^2)^{3/2} &= \int_{\pi/2}^{3\pi/2} \int_0^3 \int_0^{r^2} r^3 r\, dz\, dr\, d\theta \\ &= \int_{\pi/2}^{3\pi/2} \int_0^3 r^6\, dr\, d\theta = \int_{\pi/2}^{3\pi/2} \left.\frac{r^7}{7}\right|_0^3 d\theta \\ &= \int_{\pi/2}^{3\pi/2} \frac{3^7}{7}\, d\theta = \frac{3^7\pi}{7}.\end{aligned}$$

Alternatively, the triple intergral can be set up:

$$\int_{\pi/2}^{3\pi/2} \int_0^9 \int_{\sqrt{z}}^3 r^3 r\, dr\, dz\, d\theta.$$

3. Define $u = x+y$ and $v = x-y$. Then the Jacobian of the transformation is:

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\frac{\partial(u,v)}{\partial(x,y)}\right|^{-1} = \left|\begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array}\right|^{-1} = |-2|^{-1} = \frac{1}{2}.$$

Then

$$\begin{aligned} I &= \int_2^5 \int_1^6 \frac{1}{2} v(uv)^{1/5}\, dv\, du = \int_2^5 \int_1^6 \frac{1}{2} u^{1/5} v^{6/5}\, dv\, du \\ &= \int_2^5 \left.\frac{1}{2}\frac{5}{11} u^{1/5} v^{11/5}\right|_1^6 du = \int_2^5 \frac{5}{22} u^{1/5}\left(6^{11/5}-1\right) du \\ &= \frac{5}{22}\frac{5}{6}\left(6^{11/5}-1\right) u^{6/5}\Big|_2^5 = \frac{25}{132}\left(6^{11/5}-1\right)\left(5^{6/5}-2^{6/5}\right).\end{aligned}$$

4.

$$\begin{aligned}\oint_C P\, dx + Q\, dy &= \iint_R (Q_x - P_y)\, dA = \int_{-1}^1 \int_{-1}^1 \left(3x^2 - e^{4x}\right) dx\, dy \\ &= \int_{-1}^1 \left.\left(x^3 - \frac{e^{4x}}{4}\right)\right|_{-1}^1 dy = \int_{-1}^1 \left(2 - \frac{e^4}{4} + \frac{e^{-4}}{4}\right) dy \\ &= 2\left(2 - \frac{e^4}{4} + \frac{e^{-4}}{4}\right) = 4 - \frac{e^4}{2} + \frac{e^{-4}}{2}.\end{aligned}$$

5. div $\vec{F} = \nabla \cdot \vec{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \cdot \langle xyz, 0, -x^3 y \rangle = yz$.

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & 0 & -x^3 y \end{vmatrix} = \langle -x^3, 3x^2 y + xy, -xz \rangle.$$

6. a) $\frac{\partial}{\partial y}(ye^{xy} + \sin x) = e^{xy} + xye^{xy}$ and $\frac{\partial}{\partial x} xe^{xy} = e^{xy} + xye^{xy}$. The two partials are equal, so the field is conservative.

b) Let C be parameterized $x(t) = x_1 t, y(t) = y_1 t, 0 \le t \le 1$. Then

$$\begin{aligned} \int_C \vec{F} \cdot \vec{T}\, ds &= \int_C \vec{F} \cdot \vec{r'(t)}\, dt \\ &= \int_0^1 \left(\left((y_1 t) e^{x_1 y_1 t^2} + \sin(x_1 t) \right) x_1 + \left(x_1 t + e^{x_1 y_1 t^2} \right) y_1 \right) dt \\ &= \int_0^1 \left(2x_1 y_1 t e^{x_1 y_1 t^2} + x_1 \sin(x_1 t) \right) dt \\ &= \left(e^{x_1 y_1 t^2} - \cos(x_1 t) \right) \Big|_0^1 \\ &= e^{x_1 y_1} - \cos x_1 - (1 - 1). \end{aligned}$$

Therefore $f(x, y) = e^{xy} - \cos x$.