Lit Sheet VC.08, VC.09, VC.10 Solutions

VC.08 2)

$$\begin{aligned}
&\int_0^6 \int_0^{6-x} \int_0^{4-2(x+y)/3} (x+z)dzdydx \\
&= \int_0^6 \int_0^{6-x} \left(xz + \frac{z^2}{2} \right)\Big|_0^{4-2(x+y)/3} \\
&= \int_0^6 \int_0^{6-x} \left(8 + \frac{4x}{3} - \frac{4x^2}{9} - \frac{8y}{3} - \frac{2xy}{9} - \frac{2y^2}{9} \right) dydx \\
&= \int_0^6 \left(8y + \frac{4xy}{3} - \frac{4x^2y}{9} - \frac{4y^2}{3} - \frac{xy^2}{9} - \frac{2y^3}{27} \right)\Big|_0^{6-x} dx \\
&= \int_0^6 \left(16 + 4x - \frac{8x^2}{3} + \frac{7x^3}{27} \right) dx \\
&= \left(16x + 2x^2 - \frac{8x^3}{9} + \frac{7x^4}{108} \right)\Big|_0^6 \\
&= 60.
\end{aligned}$$

8) Let $u = z-2x$, $v = y-x$, and $w = y+2x$. Then $0 \le u \le 3$, $0 \le v \le 2$, and $0 \le w \le 4$. From the definition of v and w we see that $x = (w-v)/3$ and $y = (2v+w)/3$. Then $z = u + 2x = u + 2(w-v)/3$. We compute the fudge factor:

$$\left\| \begin{matrix} 0 & -1/3 & 1/3 \\ 0 & 2/3 & 1/3 \\ 1 & -2/3 & 2/3 \end{matrix} \right\| = \frac{1}{3}.$$

Note that the easiest way to compute this determinant is to work along the first column. Then there is only one 2×2 determinant to evaluate. Alternatively, you can create a determinant with a u-row, v-row, and w-row, and then take the reciprocal of the absolute value of the determinant. Either way you get $1/3$.

Now we set up and compute the integral.

$$\begin{aligned}\frac{1}{3}\int_0^4\int_0^2\int_0^3\left(\frac{2v+w}{3}\right)dudvdw &= \frac{1}{3}\int_0^4\int_0^2 (2v+w)\,dvdw\\ &= \frac{1}{3}\int_0^4 \left(v^2+vw\right)\Big|_0^2 dw\\ &= \frac{1}{3}\int_0^4 (4+2w)\,dw\\ &= \frac{1}{3}\left(4w+w^2\right)\Big|_0^4\\ &= \frac{32}{3}.\end{aligned}$$

VC.09 6) $\frac{\pi}{6} \leq s \leq \frac{\pi}{3}$ and $0 \leq t \leq 2\pi$. Since it's not precise, the bounds on s are flexible, as long as s_{low} is sufficiently greater than 0 and s_{high} is sufficiently less than $\pi/2$ and sufficiently greater than s_{low}.

13) Note that $x^2+y^2+z^2=r^2$, and recall that the fudge factor for spherical coordinates is $r^2\sin s$. Now we set up the integral and crunch some numbers.

$$\begin{aligned}\int_0^{2\pi}\int_0^{\pi}\int_0^{a} r^4\sin s\,drdsdt &= \int_0^{2\pi}\int_0^{\pi}\frac{a^5\sin s}{5}dsdt\\ &= \int_0^{2\pi}\frac{-a^5\cos s}{5}\Big|_0^{\pi} dt\\ &= \int_0^{2\pi}\frac{2a^5\sin s}{5}dt\\ &= \frac{4a^5\pi}{5}.\end{aligned}$$

VC.10 1) $div(Field(x,y,z)) = \frac{\partial m}{\partial x}+\frac{\partial n}{\partial y}+\frac{\partial p}{\partial z}$.
If $div(Field(x,y,z)) > 0$, then (x,y,z) is a source, and if $div(Field(x,y,z)) < 0$, then (x,y,z) is a sink.

2) $div(Field(x,y,z)) = -1-2-15z^2 = -(3+15z^2) < 0$. Since the divergence is always negative, every point is a sink, so the flow of $Field(x,y,z)$ across any closed surface is inward.

3) $div(Field(x, y, z)) = -6x + 2 + 4 = 6 - 6x$. So (x, y, z) is a source if $x < -1$ and is a sink if $x > -1$. Since $2 \leq x \leq 6$ on C_1, the net flow of $Field(x, y, z)$ across C_1 is inward. Since $-5 \leq x \leq -1$ on C_2, the net flow of $Field(x, y, z)$ across C_2 is outward.