Math 242 **December 13, 2000**
Final Exam **Bruce Berndt**

Numero deus impare gaudet (God in number odd rejoices.) Virgil

SHOW ALL WORK. INDICATE ALL REASONING.

(1) (6) a. If $\mathbf{a}$ and $\mathbf{b}$ are vectors such that $\mathbf{a} \times \mathbf{b} = \mathbf{0}$, what can you say about $\mathbf{a}$ and $\mathbf{b}$? Justify your answer.

b. Suppose that $\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{a}$. What can you say about $\mathbf{a}$ and $\mathbf{b}$? Justify your answer.

c. Is the following statement true or false?

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = \mathbf{0}.$$

Justify your answer

(2) (10) a. Find a vector orthogonal to the plane containing the points $P(2,0,-3)$, $Q(3,1,0)$, and $R(5,2,2)$.
b. Find the area of the triange PQR.

(3) (12) Find the equation of the plane that passes through the point $(1, 2, 3)$ and contains the lines $x = 3t$, $y = 1 + t$, and $z = 2 - t$.

(4) (8) Write the equation

$$z = x^2 - y^2$$

in (a) cylindrical coordinates; (b) spherical coordinates. In each case simplify your answer as much as possible.

(5) (8) Sketch the surface $y^2 = 2x^2 + 3z^2$. What are the traces of the curves in the planes $x = k$, $y = k$, and $z = k$, where k is an arbitrary constant.

(6) (15) Find the unit tangent vector, the unit normal vector, and the curvature of the curve $\mathbf{r}(t) = \tan t\ \mathbf{i} + \sin t\ \mathbf{j} + \cos t\ \mathbf{k}$ at the point $(0, 0, 1)$.

(7) (10) Find the equation of the tangent plane to

$$z = e^x \ln y$$

at the point $(3, 1, 0)$.

(8) (16) Let

$$z = f(x, y) = \ln(x^2 + y^3).$$

a. Find $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial x^2}$, and $\frac{\partial^2 f}{\partial y^2}$.

b. If

$$x = \sin t \qquad \text{and} \qquad y = \cos t,$$

use the chain rule to find

$$\frac{\partial f}{\partial t} \qquad \text{and} \qquad \frac{\partial^2 f}{\partial t^2}.$$

You do not have to simplify your answers.

(9) (12) a. Find the directional derivative of

$$w = f(x, y, z) = z^3 - x^2 y$$

at $(1, 6, 2)$ in the direction $3\mathbf{i} + 4\mathbf{j} + 12\mathbf{k}$.
b. Find the maximal value of $D_{\mathbf{u}} f(1, 6, 2)$.

(10) (13) Use the second derivative test to find all local maxima and minima for

$$f(x, y) = e^{4y - x^2 - y^2}.$$

(11) (13) Use Lagrange multipliers to find three positive numbers x, y, and z satisfying

$$x + y + z = 100$$

such that $x^a y^b z^c$ is a maximum. Here a, b, and c are positive numbers. Thus, your answer will be in terms of a, b, and c.

(12) (12) Calculate

$$\int\int_R e^{y^2}\, dA,$$

where R is the triangular region with vertices $(0,0), (1,1)$, and $(0,1)$.

(13) (13) Let $f(x,y)$ be a continuous function defined over a region R bounded by the lines $y = x+1, x = 2, y = 0$, and $xy = 2$. Indicate how you would use iterated integrals in 2 ways to evaluate

$$\int\int_R f(x,y)dA.$$

In otherwords, set up the integral(s) first integrating with respect to x and then with respect to y. Then repeat the process, integrating first with respect to y and then with respect to x.

(14) (13) Find the volume of the solid that lies within the sphere $\rho = 2\cos\phi$ but above the cone $\phi = \pi/4$.

(15) (13) Find the area of the planar region inside the circle $r = 4\sin\theta$ and outside the circle $r = 2$.

(16) (13) Find

$$\iiint_E z\,dV,$$

where E is the volume in the first octant bounded by the cylinder $y^2 + z^2 = 1$ and the plane $x + y = 2$.

(17) (13) Use a change of variables to evaluate the integral

$$\int\int_R 4xy\,dA,$$

where R is the region bounded by the straight lines

$$y-x=-1, \qquad y-x=1, \qquad x+y=1, \qquad x+y=4.$$

Math 242 Final Exam, Dec. 13, 2000 Solutions ①

1.a. $a \times b = |a||b| \sin\theta = 0 \Rightarrow \theta = 0$, or π, i.e., a and b are parallel.

b. Since $a \times b = -b \times a$, we must have $a \times b = 0$, i.e., a and b are parallel.

c. $a \times b$ is $\perp$ to the plane of a and b. In particular $a \times b \perp a$, so the statement is true.

2.a. $\vec{PQ} = +i + j + 3k$, $\vec{PR} = 3i + 2j + 5k$

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} i & j & k \\ 1 & 1 & 3 \\ 3 & 2 & 5 \end{vmatrix} = -i + 4j - k$$ is $\perp$ to the plane of $\vec{PQ}$ and $\vec{PR}$.

b. Area of $\Delta PQR = \frac{1}{2}|-i+4j-k| = \frac{1}{2}\sqrt{1+16+1} = \frac{3}{2}\sqrt{2}$

See Page 2 for the solution of Problem 3.

In Problem 4, the eq. was mistyped as $x = x^2 - y^2$

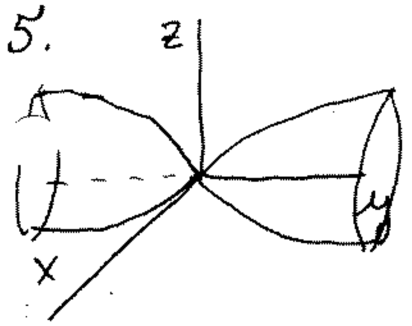
4. a. $z = r^2\cos^2\theta - r^2\sin^2\theta = r^2(\cos^2\theta - \sin^2\theta) = r^2\cos 2\theta$

b. $\rho\cos\phi = \rho^2\cos^2\theta\sin^2\phi - \rho^2\sin^2\theta\sin^2\phi$

$= \rho^2\sin^2\phi(\cos^2\theta - \sin^2\theta) = \rho^2\sin^2\phi\cos 2\theta$

$\cos\phi = \rho\sin^2\phi\cos 2\theta$

5.



$x = k$, $y^2 = 2k^2 + 3z^2$ hyperbolas

$y = k$, $k^2 = 2x^2 + 3z^2$ ellipses

$z = k$, $y^2 = 2x^2 + 3k^2$ hyperbolas

6. a. $r'(t) = \sec^2 t\, i + \cos t\, j - \sin t\, k$

(0,0,1) corresponds to $t=0$

$r'(0) = i + j$, $T(0) = \frac{1}{\sqrt{2}}(i+j)$

b. $r''(t) = 2\sec t \cdot \sec t \tan t\, i - \sin t\, j - \cos t\, k$

$r''(0) = -k$, which is a unit vector

c. $r'(0) \times r''(0) = \begin{vmatrix} i & j & k \\ 1 & 1 & 0 \\ 0 & 0 & -1 \end{vmatrix} = -i + j$

$|r'(0) \times r''(0)| = \sqrt{2}$, $|r'(0)| = \sqrt{2}$

$\therefore\ K(0) = \dfrac{\sqrt{2}}{2\sqrt{2}} = \dfrac{1}{2}$

3. The points $P(1,2,3)$ and $Q(0,1,2)$ lie in the plane. The vector $\vec{QP} = i + j + k$ and the vector $a = 3i + j - k$ lie in the plane.

$\vec{QP} \times a = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 3 & 1 & -1 \end{vmatrix} = -2i + 4j - 2k$ is a normal to the plane

$\therefore\ -2(x-1) + 4(y-2) - 2(z-3) = 0$

$2x - 4y + 2z = 0$ or $x - 2y + z = 0$.

7. $\frac{\partial z}{\partial x} = e^x \ln y$, $\frac{\partial z}{\partial y} = e^x/y$.

$\therefore\ z - 0 = 0\cdot(x-3) + e^3(y-1)$, i.e. $z = e^3(y-1)$.

8\. a. $\frac{\partial f}{\partial x} = \frac{2x}{x^2+y^3}$, $\frac{\partial f}{\partial y} = \frac{3y^2}{x^2+y^3}$, $\frac{\partial^2 f}{\partial x^2} = \frac{2y^3-2x^2}{(x^2+y^3)^2}$

$\frac{\partial^2 f}{\partial x \partial y} = \frac{-3y^2 \cdot 2x}{(x^2+y^3)^2} = \frac{-6xy^2}{(x^2+y^3)^2}$, $\frac{\partial^2 f}{\partial y^2} = \frac{6yx^2-3y^4}{(x^2+y^3)^2}$

b. $\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} = \frac{2x}{x^2+y^3}\cos t + \frac{3y^2}{x^2+y^3} \cdot -\sin t$

$\frac{\partial^2 f}{\partial t^2} = \left(\frac{2}{x^2+y^3} - \frac{4x^2}{(x^2+y^3)^2}\right)\cos^2 t + \frac{6xy^2}{(x^2+y^3)^2}\sin t\cos t$

$- \frac{2x}{x^2+y^3}\sin t$

$+ \frac{6xy^2}{(x^2+y^3)^2}\sin t\cos t + \left(\frac{6y}{x^2+y^3} - \frac{9y^4}{(x^2+y^3)^2}\right)\sin^2 t$

$- \frac{3y^2}{x^2+y^2}\cos t$

9\. a. $\frac{\partial f}{\partial x} = -2xy$, $\frac{\partial f}{\partial y} = -x^2$, $\frac{\partial f}{\partial z} = 3z^2$

$|3i+4j+12k| = \sqrt{9+16+144} = \sqrt{169} = 13$

$D_u f(1,6,2) = -12 \cdot \frac{3}{13} - 1 \cdot \frac{4}{13} + 12 \cdot \frac{12}{13} = \frac{104}{13} = 8$

b. $\max D_u f(1,6,2) = |\text{grad} f| = \sqrt{144+1+144} = \sqrt{289} = 17$

10\. $f(x,y) = e^{4y-x^2-y^2}$

$f_x = -2x e^{4y-x^2-y^2}$, $f_y = (4-2y)e^{4y-x^2-y^2}$

$f_{xx} = (4x^2-2)e^{4y-x^2-y^2}$, $f_{yy} = [(4-2y)^2-2]e^{4y-x^2-y^2}$

$= (14-16y+4y^2)e^{4y-x^2-y^2}$

$f_{xy} = -2x(4-2y)e^{4y-x^2-y^2}$

$f_x = 0 \Rightarrow x = 0$, $f_y = 0 \Rightarrow y = 2$. $(0,2)$ crit. pt.

$D(0,2) = (-2e^4)(-2e^4) - 0^2 = 4e^8 > 0$.

$f_{xx}(0,2) < 0 \quad \therefore f(0,2) = e^4$ is a local max.

11. Let $f(x,y,z) = x^a y^b z^c$, $g(x,y,z) = x+y+z = 100$.

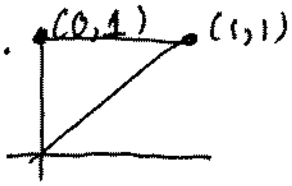
$$\text{grad} f = a x^{a-1} y^b z^c i + b x^a y^{b-1} z^c j + c x^a y^b z^{c-1} = \lambda(i+j+k),$$

$$\therefore \lambda = a x^{a-1} y^b z^c = b x^a y^{b-1} z^c = c x^a y^b z^{c-1}$$

or $$ayz = bxz = cxy \Rightarrow y = \frac{b}{a}x, \quad z = \frac{c}{a}x$$

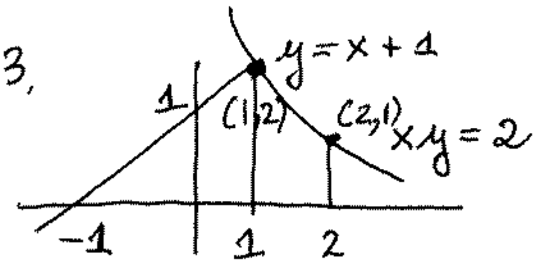
$$\therefore x+y+z = x + \frac{b}{a}x + \frac{c}{a}x = 100, \quad (a+b+c)x = 100a$$

$$\therefore x = \frac{100a}{a+b+c}, \quad y = \frac{100b}{a+b+c}, \quad z = \frac{100c}{a+b+c} \text{ (by symmetry)}$$

12. 

$$\iint_R e^{y^2}\,dA = \int_0^1 \int_0^y e^{y^2}\,dx\,dy = \int_0^1 e^{y^2} y\,dy$$

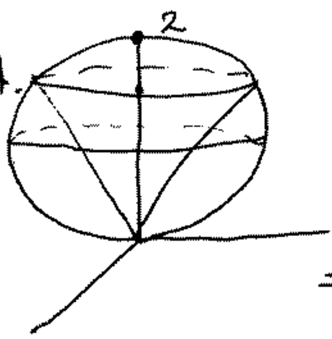
$$= \frac{1}{2} e^{y^2}\Big|_0^1 = \frac{1}{2}(e-1)$$

13. 

$$x(x+1) = 2 \Rightarrow x^2 + x - 2 = (x+2)(x-1) = 0 \Rightarrow x = 1, -2$$

$$\int_{-1}^{1}\int_0^{x+1} f(x,y)\,dy\,dx + \int_1^2 \int_0^{2/x} f(x,y)\,dy\,dx$$

$$\int_0^1 \int_{y-1}^{2} f(x,y)\,dx\,dy + \int_1^2 \int_{y-1}^{2/y} f(x,y)\,dx\,dy$$

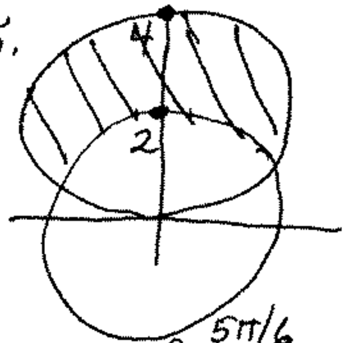
14. 

$$V = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{2\cos\phi} \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/4} \frac{1}{3}(2\cos\phi)^3 \sin\phi\,\underset{d\phi}{d\theta} = 2\pi \cdot \frac{8}{3} \cdot -\frac{\cos^4\phi}{4}\Big|_0^{\pi/4}$$

$$-\frac{\cdots}{3}\left(1-\frac{1}{4}\right) = \frac{4\pi}{3}\cdot\frac{3}{4} = \pi$$

5.

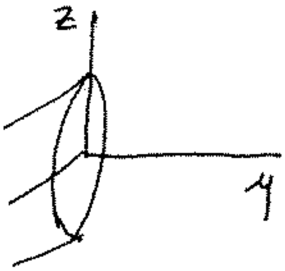


$$2 = 4\sin\theta \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta =$$

$$A = \int_{\pi/6}^{5\pi/6}\int_{2}^{4\sin\theta} r\,dr\,d\theta = \int_{\pi/6}^{5\pi/6} \frac{1}{2}r^2\Big|_2^{4\sin\theta} d\theta$$

$$= \frac{1}{2}\int_{\pi/6}^{5\pi/6}(16\sin^2\theta - 4)\,d\theta = \int_{\pi/6}^{5\pi/6}[4(1-\cos 2\theta)-2]\,d\theta$$

$$= \int_{\pi/6}^{5\pi/6}(2-4\cos 2\theta)\,d\theta = 2\theta - 2\sin 2\theta\Big|_{\pi/6}^{5\pi/6} = \frac{4\pi}{3} + 2\sqrt{3}$$



$$\iiint_E z\,dV = \int_0^1\int_0^{\sqrt{1-y^2}}\int_0^{2-y} z\,dx\,dz\,dy$$

$$= \int_0^1\int_0^{\sqrt{1-y^2}} z(2-y)\,dz\,dy = \frac{1}{2}\int_0^1 (1-y^2)(2-y)\,dy$$

$$\int_0^1 (2-y-2y^2+y^3)\,dy = \frac{1}{2}\left(2-\frac{1}{2}-\frac{2}{3}+\frac{1}{4}\right) = \frac{13}{24}$$

Sol.

$$\int_E z\,dV = \int_0^{\pi/2}\int_0^1\int_0^{2-y} r\cos\theta\, r\,dx\,dr\,d\theta = \int_0^{\pi/2}\int_0^1 r^2\cos\theta(2-r\sin \ldots\, dr\,d\ldots$$

$$= \int_0^{\pi/2}\int_0^1 (2r^2\cos\theta - r^3\cos\theta\sin\theta)\,dr\,d\theta$$

$$= \frac{2}{3}r^3\Big|_0^1 \sin\theta\Big|_0^{\pi/2} - \frac{r^4}{4}\Big|\frac{\sin^2\theta}{2}\Big|^{\pi/2} = \frac{2}{3} - \frac{1}{8} = \frac{16-3}{24} = \frac{13}{24}$$

17. Let $u = y - x$, $v = y + x$. Then,

$$x = \frac{1}{2}(v-u), \quad y = \frac{1}{2}(u+v)$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$

$$\iint 4xy\,dA = \frac{1}{2}\int_1^4 \int_{-1}^{1} (v^2 - u^2)\,du\,dv = \frac{1}{2}\int_1^4 \left(v^2 u - \frac{u^3}{3}\right)\Big|_{-1}^{1} dv$$

$$= \frac{1}{2}\int_1^4 \left(2v^2 - \frac{2}{3}\right) dv = \frac{1}{2}\left(\frac{2v^3}{3} - \frac{2}{3}v\right)\Bigg|_1^4 = \frac{1}{2}\left(\frac{128}{3} - \frac{8}{3}\right) = \frac{120}{6} = 20$$