Sample Final Solutions

1) $\vec{x}\cdot\vec{y} = 2-5+3=0$, so $\vec{x}$ and $\vec{y}$ are perpendicular.

2) The normal vector of the plane is

$$\vec{x}\times\vec{y} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -5 & 1 \\ 2 & 1 & 3 \end{vmatrix} = \{-17,-1,11\}.$$

Since the plane contains the origin $\{x,y,z\}$ is an arbitrary vector in the plane, so

$$\{x,y,z\}\cdot\{-17,-1,11\} = -17x - y + 11z = 0$$

is the equation of the plane.

3) $\vec{r}(t) = \{3,-1,4\} + t\{-17,-1,11\} = \{3-17t, -1-t, 4+11t\}$.

4) a) Wind flow, fluid flow, an electromagnetic field, etc.
b) No. Vectors in a vector field are tangent to any trajectory in the field, so if two trajectories were to cross, there would be two different vectors at the same point, which is impossible.

5) $\frac{\partial n}{\partial x} - \frac{\partial m}{\partial y} = xy$. The easiest way to find m and n is to let $m(x,y)=0$, then integrate both sides with respect to x to get $n(x,y) = x^2y/2$.

Now $x(t) = 5\cos t$ and $y(t) = 2\sin t$, so by the Gauss-Green Theorem,

$$\begin{aligned}\int_a^b (m(x(t),y(t))x'(t) + n(x(t),y(t))y'(t))dt &= \int_0^{2\pi} 25\cos^2 t\sin t\cdot 2\cos t dt \\ &= 50\int_0^{2\pi}\cos^3 t\sin t dt.\end{aligned}$$

6)

$$\begin{aligned}\int_{-\pi}^{2\pi}\int_{-\pi}^{2\pi}(3+\sin(2x)+\cos(3y))dxdy &= \int_{-\pi}^{2\pi}\left(3x - \frac{\cos(2x)}{2} + x\cos(3y)\right)\Bigg|_{-\pi}^{2\pi} \\ &= \int_{-\pi}^{2\pi}(9\pi + 3\pi\cos(3y))dy \\ &= 9\pi y + \pi\sin(3y)|_{-\pi}^{2\pi} \\ &= 27\pi^2 - \pi.\end{aligned}$$

7) Since $f(x_0, y_0)$ is a minimum, the vectors of the gradient field will be pointing away from (x_0, y_0), so the flow across C is inside to outside.

8) First note that the divergence of $Field(x, y)$ is $2(x + y)$. So the net flow of the field across C is

$$\begin{aligned}\int_{-2}^{1}\int_{0}^{1} 2(x+y)dydx &= \int_{-2}^{1}(2xy+y^2)|_0^1 dx\\ &= \int_{-2}^{1}(2x+1)dx\\ &= (x^2+x)|_{-2}^{1} = 2-2 = 0.\end{aligned}$$

So there is no net flow.

9) The gradient of f points in the direction of greatest increase of a function at a point. The greadient is $\nabla f = \{2xz + y^3, 3xy^2, x^2\}$, so at the point $\{1, 3, -1\}$, $\nabla f(1, 3, -1) = \{25, 27, 1\}$.

10) a) $rot(Field(x, y)) = \frac{\partial n}{\partial x} - \frac{\partial m}{\partial y} = e^x \sin y + e^x \sin y$.

b) I wouldn't try this part. I made a mistake in defining the field. My gut feeling is that it's 0, but to compute it using $rot(Field)$ will be messy.

11) $\Delta f = div(\nabla f) = 0$, so the divergence of the gradient field of f is zero, meaning there is no flow across any closed curve except possibly for curves around singularities. Since f has no singularities, the net flow is always 0.

12) i) Across, ii) Along, iii) Along, iv) Along, v) Across, vi) Along, vii) Across.

13)

$$\begin{aligned}\iint_R 3y\,dx\,dy &= \int_0^{2\pi}\int_0^1 3r^2\sin t\,dr\,dt\\ &= \int_0^{2\pi} r^3\sin t\Big|_0^1 dt\\ &= \int_0^{2\pi}\sin t\,dt\\ &= -\cos t\Big|_0^{2\pi} = 0.\end{aligned}$$

14) Let $u = y - x$ and $v = y + x$. Solving for x and y we get $x = \frac{v-u}{2}$ and $y = \frac{u+v}{2}$, and it follows that the fudge factor is 1/2. So:

$$\begin{aligned}\iint_R y\,dx\,dy &= \int_0^2\int_0^2 \frac{u+v}{4}du\,dv\\ &= \int_0^2 \left(\frac{u^2}{8}+\frac{uv}{4}\right)\Big|_0^2 dv\\ &= \int_0^2\left(\frac{1}{2}+\frac{v}{2}\right)dv\\ &= \frac{v}{2}+\frac{v^2}{4}\Big|_0^2\\ &= 2.\end{aligned}$$

15) The transformation is $x(r,t) = ar\cos t$, $y(r,t) = br\sin t$. The fudge factor is abr, and the integral is:

$$\begin{aligned}\int_0^{2\pi}\int_0^1 abr\,dr\,dt &= \int_0^{2\pi}\frac{abr^2}{2}\Big|_0^1 dt\\ &= \int_0^{2\pi}\frac{ab}{2}dt\\ &= ab\pi.\end{aligned}$$

16)

$$\int_0^2\int_0^{4-2z}\int_0^{8-2y-4z} x\,dx\,dy\,dz.$$

17) Let $u = z - x$, $v = y + x$, and $w = y - x$. Then $x = \frac{v-w}{2}$, $y = \frac{v+w}{2}$, and $z = u + \frac{v-w}{2}$, and we see that the fudge factor is 1/2. The volume of the parallelepiped is:

$$\int_{-2}^{2} \int_{-1}^{1} \int_{0}^{2} \frac{1}{2} dudvdw = 4 \cdot 2 \cdot 2 \cdot \frac{1}{2} = 8.$$

18) Note that the z-coordinate of the surface should just be s. Assume that the fudge factor is the same.

$$\int_{0}^{1} \int_{0}^{2\pi} \int_{-1}^{1} 2(1 - s^4)udsdtdu.$$

19) The divergence of the field is $2x + 2y - 4yz$, so the flow across the surface of the box is

$$\begin{aligned}
\int_{0}^{2} \int_{-1}^{3} \int_{0}^{3} (2x + 2y - 4yz)dzdydx &= \int_{0}^{2} \int_{-1}^{3} (2xz + 2yz - 2yz^2)|_0^3 dydx \\
&= \int_{0}^{2} \int_{-1}^{3} (6x + 6y - 18y)dydx \\
&= \int_{0}^{2} (6xy + 3y^2 - 9y^2)|_{-1}^3 dx \\
&= \int_{0}^{2} (18x + 27 - 81) - (6x + 3 - 9)dx \\
&= \int_{0}^{2} (12x - 48)dx \\
&= 6x^2 - 48x|_0^2 = 24 - 96 = -72.
\end{aligned}$$

The flow is outside to inside.

20) Since the only possible source or sink is the singularity itself, I would simplify the surface to a sphere, which has normal vector $\{x, y, z\}$ at the point $\{x, y, z\}$. In two dimensions, we dotted the field vector with the normal vector of the curve, and here it is no different. We dot the $\{x, y, z\}$ with the field vector, and if the result is a constant, then we have the answer. Otherwise, we parametrize the surface of a sphere or radius 1, say: $\{x(s,t), y(s,t), z(s,t)\} = \{s \cos t, s \sin t, s\}$, where $-1 \leq s \leq 1$ and $0 \leq t \leq 2\pi$, and integrate the dot product.