Sample Test 2 Solutions

1. I would rather integrate first with respect to y since the region R is vertically convex, that is, the top and bottom boundaries are given as functions of x.

Before calculating the integral, we need to determine the bounds of integration. The bounds for y are given. The bounds for x are determined by where the two boundary curves intersect.

$$x(x-1) = 2x(1-x)$$

By inspection, we see that $x = 0$ and $x = 1$ are the two solutions.

$$\begin{aligned}
\iint_R (x+y)dydx &= \int_0^1 \int_{x(x-1)}^{2x(1-x)} (x+y)dydx \\
&= \int_0^1 (xy+y^2)\Big|_{x(x-1)}^{2x(1-x)} dx \\
&= \int_0^1 (2x^2(1-x) + 4x^2(1-x)^2 - x^2(x-1) - x^2(x-1)^2)dx \\
&= \int_0^1 (3x^2(1-x) + 3x^2(1-x)^2)dx \\
&= \int_0^1 (6x^2 - 9x^3 + 3x^4)dx \\
&= \left(2x^3 - \frac{9}{4}x^4 + \frac{3}{5}x^5\right)\Bigg|_0^1 \\
&= 2 - \frac{9}{4} + \frac{3}{5} = \frac{7}{20}.
\end{aligned}$$

2. TYPO! As given, this integral will be impossible to integrate if we switch the order of integration. Switching the order will give us $\int_0^1 \int_x^1 e^{-x^2} dxdy$, so

the only way to evaluate it is to go with what's given.

$$\begin{aligned}\int_0^1 \int_0^x e^{-x^2} dy dx &= \int_0^1 e^{-x^2} y\Big|_0^x dx \\ &= \int_0^1 xe^{-x^2} dx \\ &= -\frac{1}{2}e^{-x^2}\Big|_0^1 = 1 - e^{-1}.\end{aligned}$$

3. Parametrize the circle via $x(t) = 2\cos t$, $y(t) = 2\sin t$. Then the flow across the circle is:

$$\begin{aligned}&\oint_C \{-n(x(t), y(t)), m(x(t), y(t))\} \cdot \{dx, dy\} dt \\ &= \int_0^{2\pi} \{-y(t), x(t)\} \cdot \{-2\sin t, 2\cos t\} dt \\ &= \int_0^{2\pi} \{-2\sin t, 2\cos t\} \cdot \{-2\sin t, 2\cos t\} dt \\ &= \int_0^{2\pi} \left(4\sin^2 t + 4\cos^2 t\right) dt = 8\pi.\end{aligned}$$

Thus the flow is outward.

4. Parametrize the circle via $x(t) = \cos t$, $y(t) = \sin t$. Then the flow along the circle is:

$$\begin{aligned}&\oint_C \{m(x(t), y(t)), n(x(t), y(t))\} \cdot \{dx, dy\} dt \\ &= \int_0^{2\pi} \{x(t), y(t)\} \cdot \{-\sin t, \cos t\} dt \\ &= \int_0^{2\pi} \{\cos t, \sin t\} \cdot \{-\sin t, \cos t\} dt \\ &= \int_0^{2\pi} (-\sin t \cos t + sint \cos t)\, dt = 0.\end{aligned}$$

Thus there is no flow around the circle.

6. Note that we need to find $m(x, y)$ and $n(x, y)$ such that

$$\frac{\partial n}{\partial x} - \frac{\partial m}{\partial y} = 2xy^2.$$

The easiest way to do this is to let $m(x, y) = 0$ and solve for $n(x, y)$ via integration. So $m(x, y) = 0$ and $n(x, y) = x^2y^2$ define our field. We now set up the integral:

$$\int_0^{2\pi} -108 \sin^3 t \cos^2 t dt.$$

You don't need to evaluate it, but it's not as bad as it looks. Think about how you would go about solving it.

7. $Field1(x, y)$ is not a gradient field, but $Field2(x, y)$ is:

$$\frac{\partial(2x + 3y)}{\partial y} = 3 \neq -1 = \frac{\partial(-x - 3y)}{\partial x}.$$

$$\frac{\partial ye^{xy}}{\partial y} = e^{xy} + xye^{xy},$$

and

$$\frac{\partial xe^{xy}}{\partial x} = e^{xy} + xye^{xy}.$$

Notice that $f(x, y) = e^{xy}$ is the function whose gradient is $Field2(x, y)$.

8. First notice that the given field is a gradient field. ($f(x, y) = xy^5$ if you're curious.) Integrals are path independent in this case, so since C_1 starts where C_2 finishes and C_2 starts where C_1 finishes, the two integrals are opposite each other.

12. The double integral measures the volume under the surface $z = f(x, y)$ over the region R.

13. Set $u(x, y) = x + y$ and $v(x, y) = x - y$. By adding the two functions, we solve for x and get $x(u, v) = \frac{u+v}{2}$, and likwise $y(u, v) = \frac{u-v}{2}$. Our "fudge factor" is

$$\left\| \begin{matrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{matrix} \right\| = \left| -\frac{1}{4} - \frac{1}{4} \right| = \frac{1}{2}.$$

On u-v paper, the integral is then:

$$\begin{aligned}
\frac{1}{2}\int_0^2\int_1^2 \frac{u+v}{2} dvdu &= \frac{1}{4}\int_0^2 \left(uv+\frac{v^2}{2}\right)\Big|_1^2 du \\
&= \frac{1}{4}\int_0^2 \left(2u+2-u-\frac{1}{2}\right) du \\
&= \frac{1}{4}\left(\frac{u^2}{2}+\frac{3u}{2}\right)\Big|_0^2 \\
&= \frac{1}{4}(2+3) = \frac{5}{4}.
\end{aligned}$$

14. We parametrize the elliptical region R by $x(r,t) = r\cos t$ and $y(r,t) = 2r\sin t$, where $0 \leq r \leq 1$ and $0 \leq t \leq 2\pi$. Our scale factor is then:

$$\left|\left|\begin{matrix} \cos t & -r\sin t \\ 2\sin t & 2r\cos t \end{matrix}\right|\right| = |2r\cos^t + 2r\sin^2 t| = 2r.$$

So we set up the integral:

$$\begin{aligned}
2\int_0^{2\pi}\int_0^1 re^{-r^2} drdt &= \int_0^{2\pi} -e^{-r^2}\Big|_0^1 dt \\
&= \int_0^{2\pi} (1-e^{-1})dt = 2(1-e^{-1})\pi.
\end{aligned}$$

11. i) Flow across.
ii) Flow along.
iii) Flow along.
iv) Flow along.
v) Flow across.
vi) Flow along.
vii) Flow across.