SOLUTIONS TO THE THIRD MIDTERM FOR MATH 380

1. (10 points) Evaluate $I := \int_C 2xyzdx + x^2zdy + x^2ydz$, where C is an oriented simple curve connecting $(1, 1, 1)$ to $(1, 2, 4)$.

Solution. Note that $2xyzdx + x^2zdy + x^2ydz = dF$, where $F = x^2yz$. Since F has continuous first partial derivatives everywhere in the space, the integral is independent of path and for any simple oriented curve connecting $(1, 1, 1)$ to $(1, 2, 4)$ we have

$$I = \int_{(1,1,1)}^{(1,2,4)} dF = F(1, 2, 4) - F(1, 1, 1) = 8 - 1 = 7.$$

2. (10 points) Let σ be the path given by $\sigma(t) = (t^2, t, 3)$ for $t \in [0, \frac{e^2-1}{4e}]$. Find the length l of the path.

Solution. For a path $\sigma : x = x(t), y = y(t), z = z(t)$, $t_1 \leq t \leq t_2$ we will use the following formula for its length $l(C)$:

$$l(C) = \int_{t_1}^{t_2} \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} dt.$$

Since in our case $x(t) = t^2, y(t) = t, z(t) = 3$ and $0 \leq t \leq \frac{e^2-1}{4e}$, we have

$$l = \int_0^{\frac{e^2-1}{4e}} \sqrt{4t^2+1} dt. \tag{1}$$

To compute the last integral we make the change of variables: $2t = \sinh u = \frac{e^u - e^{-u}}{2}$. Then

$$4t^2 + 1 = \left(\frac{e^u - e^{-u}}{2}\right)^2 + 1 = \frac{e^{2u} - 2 + e^{-2u}}{4} + 1 = \left(\frac{e^u + e^{-u}}{2}\right)^2 = \cosh^2 u,$$

$$2dt = \frac{e^u + e^{-u}}{2} du = \cosh u du,$$

$t = 0$ implies $u = 0$ and $2t = \frac{e^2-1}{2e} = \frac{e-e^{-1}}{2} = \sinh 1$, i.e., $t = \frac{e^2-1}{4e}$ implies $u = 1$. Thus, (1) becomes

$$l = \frac{1}{2}\int_0^1 \cosh^2 u du = \frac{1}{8}\int_0^1 e^{2u} + 2 + e^{-2u} du = \frac{1}{8}\left(\frac{1}{2}e^2 + 2 - \frac{1}{2}e^{-2}\right) = \frac{1}{8}\sinh 2 + \frac{1}{4}.$$

3. (10 points) Let S be the closed surface that consists of the hemisphere $x^2+y^2+z^2=1$, $z \geq 0$, and its base $x^2+y^2 \leq 1$, $z=0$. Let $\mathbf{E}$ be the electric field defined by $\mathbf{E}(x,y,z) = 2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k}$. Find the electric flux across S (i.e., the flux of $\mathbf{E}$ across S), if $\mathbf{n}$ is the outer normal.

Solution. By definition, the flux of $\mathbf{E}$ across S is $\int\int_S E_n d\sigma$ which by the Divergence theorem equals $\int\int_R\int \text{div}\mathbf{E}\, dxdydz$:

$$I := \int\int_S E_n d\sigma = \int\int_R\int \text{div}\,\mathbf{E}\, dxdydz.$$

Here div $\mathbf{E}$ $= 2+2+2 = 8$ and R is the half of the unit ball. Thus,

$$I = 8\int\int_R\int dxdydz = 8 \cdot (\text{volume of } R) = 8 \cdot \frac{1}{2} \cdot \frac{4\pi}{3} = \frac{16\pi}{3}.$$

4. (10 points) Evaluate the integral

$$I = \oint_C \frac{-ydx + xdy}{x^2 + y^2}$$

on the ellipse $C : \frac{x^2}{2} + \frac{y^2}{3} = 1$ oriented counterclockwise.

Solution. Let $P = -y/(x^2 + y^2)$, $Q = x/(x^2 + y^2)$. Note that $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$. Since P and Q are discontinuous only at $(0, 0)$, they are continuous in the region between the ellipse $C : \frac{x^2}{2} + \frac{y^2}{3} = 1$ and the unit circle S. Hence by Green's theorem $\oint_C Pdx + Qdy = \oint_S Pdx + Qdy$, where S is oriented counterclockwise. Parameterizing S in the usual way, namely $x = \cos t$, $y = \sin t$, $0 \leq t \leq 2\pi$, and taking into account that $x^2 + y^2 = 1$ for $(x, y) \in S$, we get

$$I = \oint_S Pdx+Qdy = \int_0^{2\pi} (-\sin t\cdot(-\sin t)+\cos t\cdot\cos t)dt = \int_0^{2\pi} (\sin^2 t+\cos^2 t)dt = \int_0^{2\pi} dt = 2\pi.$$

5. (10 points) Compute $I := \int\int_S xy d\sigma$, where S is the surface of the tetrahedron with sides $z = 0$, $y = 0$, $x + z = 1$, and $x = y$.

Solution. Let S_1 denote the side of S given by $z = 0$, let S_2 denote the side of S given by $y = 0$, let S_3 denote the side of S given by $x + z = 1$, and let S_4 denote the side of S given by $x = y$. Then $I = I_1 + I_2 + I_3 + I_4$, where $I_i := \int\int_{S_i} xy d\sigma$, $1 \leq i \leq 4$. We can now compute each integral I_i using formula (5.76) or (5.79) (pp. 308 – 309 in your textbook). Let us use formula (5.79). For S_1, we take the upper normal $\mathbf{n}_1 = (0, 0, 1)$ and $\mathbf{v}_1 = (0, 0, xy)$ so that $xy = \mathbf{v}_1 \cdot \mathbf{n}_1$. Then by (5.79), $I_1 = \int\int_{S_1} xy dx dy$ and by (5.80) we get $I_1 = \int\int_{R_{xy}} xy dx dy$, where R_{xy} is the triangle in the xy-plane with vertices $(0, 0)$, $(1, 0)$, and $(1, 1)$. Thus,

$$I_1 = \int_0^1 \int_0^x xy dx dy = \frac{1}{2} \int_0^1 x^3 dx = \frac{1}{8}.$$

For S_2, the function xy is zero on S_2, hence $I_2 = 0$.

For S_3, we take the upper normal $\mathbf{n}_3 = (1/\sqrt{2}, 0, 1/\sqrt{2})$ and $\mathbf{v}_3 = (0, 0, xy)$ so that $xy = \sqrt{2} \cdot \mathbf{v}_3 \cdot \mathbf{n}_3$. Then by (5.79), $I_3 = \sqrt{2} \int\int_{S_3} xy dx dy$ and by (5.80) we get $I_3 = \sqrt{2} \int\int_{R_{xy}} xy dx dy$, where R_{xy} is again the triangle in the xy-plane with vertices $(0, 0)$, $(1, 0)$, and $(1, 1)$. Thus,

$$I_3 = \sqrt{2} I_1 = \frac{\sqrt{2}}{8}.$$

Finally, for S_4, we take the upper normal $\mathbf{n}_4 = (1/\sqrt{2}, -1/\sqrt{2}, 0)$ and $\mathbf{v}_4 = (xy, 0, 0)$ so that $xy = \sqrt{2} \cdot \mathbf{v}_4 \cdot \mathbf{n}_4$. Then by (5.79), $I_4 = \sqrt{2} \int\int_{S_4} xy dy dz = \sqrt{2} \int\int_{S_4} y^2 dy dz$ and by (5.80) we get $I_4 = \sqrt{2} \int\int_{R_{yz}} y^2 dy dz$, where R_{yz} is the triangle in the yz-plane with vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$. Thus,

$$I_4 = \sqrt{2} \int_0^1 \int_0^{1-y} y^2 dz dy = \sqrt{2} \int_0^1 (y^2 - y^3) dy = \frac{\sqrt{2}}{12}$$

and hence $I = I_1 + I_2 + I_3 + I_4 = (5\sqrt{2} + 3)/24$.