Solutions to Sample Exam 2

1. Last exam's material. If you're curious, the limit is 0. Substitute $x = 0$ and take $y \to 0$, and then substitute $y = mx$ and take $x \to 0$.

2. $\sqrt{3(2.1)^2 + 6(1.9)^2}$ is $f(2.1, 1.9)$, where $f(x, y) = \sqrt{3x^2 + 6y^2}$. Using linear approximation, then:

$$f(2.1, 1.9) \approx f(2, 2) + df = f(2, 2) + f_x(2, 2)\Delta x + f_y(2, 2)\Delta y.$$

Find the partials, find Δx and Δy, and $f(2.1, 1.9) \approx 5.9$.

3. Draw a diagram showing the dependence of the variables w, u, v, z, s, and t. Then

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial s}.$$

So

$$\frac{\partial w}{\partial s} = \frac{3ue^t \cos s + 3v}{\sqrt{u^2 + v^2 + z^2}}.$$

4. The directional derivative $D_{\vec{u}}(f) = \nabla f \cdot \vec{u}$, where $\vec{u}$ is a unit vector. So we have:

$$\nabla f \cdot \vec{u} = \langle 2x - 2y, -2x + 6y\rangle \cdot \frac{\langle 1, 1\rangle}{\sqrt{2}} = \langle -2, 10\rangle \cdot \frac{\langle 1, 1\rangle}{\sqrt{2}} = 4\sqrt{2}.$$

5. Find $f_x = 3x^2 + 6y - 6$, $f_y = 6x + 6y - 6$, from which we find our critical points $(0, 1)$ and $(2, 1)$. Then find $f_{xx} = 6x$, $f_{xy} = f_{yx} = 6$, and $f_{yy} = 6$. The second partials determinant is then $36x - 36$, so for:
$(0, 1) : \Delta < 0$ and is a saddle point.
$(2, -1) : \Delta > 0, f_{xx} > 0$ and is a (global) minimum.
Note: You do not need to say this is a global minimum, just a minimum. Looking at the dominant terms of $f(x, y)$ tells us that as $|x|$ and $|y|$ increase, $f(x, y)$ increases without bound.

6. Step A: Find the ciritical point(s) of $f(x, y)$. You just get one: $(0, 0)$, and $f(0, 0) = 0$.

Step B: Find critical points along the 3 boundaries.

1. $y = 0$ yields one critical point - the origin.
2. $x = 0$ yields one critical point - the origin again.
3. The third boundary is $y = 1 - x$. Plug this into $f(x, y) = f(x, 1 - x) = -3x^2 + 5x - 1$. This gives the single critical point $(5/6, 1/6)$, and $f(5/6, 1/6) = 13/12$.

Step C: Check the 2 remaining corners. (We already got the origin.)
1. $f(0, 1) = -1$
2. $f(1, 0) = 1$

Compare all the corners and critical points.
Global maximum: 13/12 at $(5/6, 1/6)$
Global minimum: -1 at $(0, 1)$.

7. Find the gradients:
$\nabla f = \langle 2x, -2y, 2z \rangle$,$\nabla g = \langle 1, 1, 1 \rangle$, $\nabla f = \langle 1, -2, 3 \rangle$,
so to set up three of the equations:

$$\nabla f = \lambda \nabla g + \mu \nabla h.$$

We have 5 equations and 5 unknowns:

$$\begin{aligned} x + y + z &= 1 \\ x - 2y + 3z &= 3 \\ 2x &= \lambda + \mu \\ -2y &= \lambda - 2\mu \\ 2z &= \lambda + 3\mu \end{aligned}$$

There are a number of ways to do this. What I did is work with the last three, solving for x, y, and z, and plugged that into the first two equations, so we get the system, after some simplification:

$$\begin{aligned} \lambda + 6\mu &= 2 \\ \lambda + \mu &= 1, \end{aligned}$$

from which we find that $\lambda = 4/5$ and $\mu = 1/5$. It follows that the only solution to the five equations above is then $(x, y, z) = (1/2, -1/5, 7/10)$, and $f(1/2, -1/5, 7/10) = 7/10$. It's not immediately obvious that this point is a max, min, or saddle point, though given that $f(x, y, z) = x^2 - y^2 + z^2$, it's most likely a minimum.

To find out for sure, we parametrize the line of intersection of $g = 0, h = 0$, plug that into $f(x, y, z)$, and find critical points. The line of intersection is: $x = 5t, y = -2t, z = 1 - 3t$, so $f(x(t), y(t), z(t)) = 30t^2 - 6t + 1$. This is the value of f along the line of intersection of the two planes. Since the dominant term is $30t^2$, $7/10$ is the minimum of f subject to the two constraints.