Solutions to Sample Exam 3

1. For the tricky integration part, you'll need to use integration by parts:

$$\int_a^b u dv = uv|_a^b - \int_a^b v du.$$

For the problem, sketch the region of integration, then we set it up:

$$\int_0^\pi \int_0^{\sin x} x dy\, dx = \pi.$$

2. The region of integration is a right isosceles triangle whose top border is $y = \pi$, left border is the y-axis, and bottom/right border is $y = x$. After switching the order of integration, we get:

$$\int_0^\pi \int_0^y \frac{\sin y}{y} dx\, dy = 2.$$

3. Sketch this guy out. The bottom of the region will be $y = x^2$ and the top will be $\frac{2}{1+x^2}$. They intersect at $x = \pm 1$. (Ignore the complex solutions.) So the double integral will be

$$\int_{-1}^1 \int_{x^2}^{\frac{2}{1+x^2}} dy\, dx = \pi - \frac{2}{3}.$$

Note: $\int \frac{dx}{1+x^2} = \tan^{-1} x + C$.

4. The intersection of the cone and sphere is found by solving:

$$\begin{aligned} z = \sqrt{1 - x^2 - y^2} &= \sqrt{x^2 + y^2} \\ \sqrt{1 - r^2} &= \sqrt{r^2} \\ 1 - r^2 &= r^2 \\ 1 &= 2r^2 \\ r &= \frac{\sqrt{2}}{2} \end{aligned}$$

So the intersection is a circle of radius $\frac{\sqrt{2}}{2}$ centered at the origin. The integral is then set up:

$$\int_0^{2\pi} \int_0^{\frac{\sqrt{2}}{2}} \left(\sqrt{1-r^2} - r\right) r \, dr \, d\theta = \frac{2}{3}\left(1 - \frac{\sqrt{2}}{2}\right)\pi.$$

5. [SKIP]

6. Sketch out the region. Each of the six boxes have area 1, so for the Riemann sum we get:

$$\begin{aligned}\sum_i f(x_i^*, y_i^*)\Delta x \Delta y &= f(1,0) + f(2,0) + f(1,-1) \\ &\quad + f(2,-1) + f(1,-2) + f(2,-2) \\ &= 4 + 32 + 10 + 44 + 28 + 80 \\ &= 198.\end{aligned}$$

The actual volume,

$$\int_{-2}^{1} \int +1^3 (4x^3 + 6xy^2) dx \, dy = 312.$$

7. [SKIP]